Problem Set 2

1. For each of the following subsets of $\mathbb{C}$, determine whether or not it is open, closed, (path) connected, bounded.

(i) $A = \{z \in \mathbb{C} : Re(z^2) > 1\}$ (ii) $A = \{z \in \mathbb{C} : |Im(z)| < 2\}$

(iii) $A = \{z \in \mathbb{C}, z \neq 0 : 0 \leq \arg(z) \leq \frac{\pi}{2}\}$ (iv) $A = \{z \in \mathbb{C} : 1 < |z-2| \leq 2\}$

(v) $A = \{z \in \mathbb{C} : |2z+3| > 4\}$ (vi) $A = \{z \in \mathbb{C}, z \neq 0 : Im\left(\frac{1}{z}\right) > 1\}$

(vii) $A = \{z \in \mathbb{C} : |e^z| = 2\}$

2. Determine $Acc(A)$ (set of accumulation points), $\overline{A}$, $\mathring{A}$ and ∂A, for $A \subset \mathbb{C}$ given in (i) – (vii) of Problem 1 above.

3. Let $A \subset \mathbb{C}$ and $\overline{A} = A \cup Acc(A)$. Show that $\overline{A}$ is closed, and for any closed set B with $A \subset B$, we have $\overline{A} \subset B$.

4. Compute the following limits, or prove that it does not exist.

(i) $\lim\limits_{z \to 0} \dfrac{\overline{z}}{|z|}$ (ii) $\lim\limits_{z \to \infty} \dfrac{z+1}{z^2+2}$ (iii) $\lim\limits_{z \to 0} \dfrac{Re(z^2)}{|z|^2}$

(iv) $\lim\limits_{z \to \infty} e^z$

5. Let $P(z)$ and $Q(z)$ be two polynomials of degree n and m respectively.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$$
$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0$$

$\left(a_0, \dots, a_n; b_0, \dots, b_m \in \mathbb{C}\right)$. $a_n \neq 0$ and $b_m \neq 0$

Assume $n \leq m$ and show that
$$\lim_{z \to \infty} \frac{P(z)}{Q(z)} = \begin{cases} \dfrac{a_n}{b_n} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}.$$

6. Let $\mathcal{U} \subset \mathbb{C}$ be an open set and $z_0 \in \mathcal{U}$. Define $\mathcal{U}_1 \subset \mathcal{U}$ as
$$\mathcal{U}_1 := \left\{ w \in \mathcal{U} \text{ such that there exists a path } \gamma : [0,1] \to \mathcal{U} \text{ s.t. } \begin{array}{l} \gamma(0) = z_0 \\ \gamma(1) = w \end{array} \right\}$$
Prove that $\mathcal{U}_1$ and $\mathcal{U} \setminus \mathcal{U}_1$ are both open.

7. Let $A \subset \mathbb{C}$ be (path) connected. Show that $\overline{A}$ is also (path) connected.

8. Let $A_1, A_2 \subset \mathbb{C}$ be two path-connected subsets, such that $A_1 \cap A_2 \neq \phi$. Prove that $A_1 \cup A_2$ is again path-connected

9. Let $S = \{ z \in \mathbb{C} : \text{either } \mathrm{Re}(z) \in \mathbb{Q} \text{ or } \mathrm{Im}(z) \in \mathbb{Q} \}$. Prove that S is (path) connected

10. Prove the "squeeze thm".: Let $f, g : \overset{\times}{D}(0;1) \to \mathbb{C}$ (recall $\overset{\times}{D}(0;1) = \{ 0 < |z| < 1 \}$) be such that (i) $\lim_{z \to 0} f(z) = 0$, (ii) $\exists\, M \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}$, $0 < r < 1$ such that $|g(z)| < M$, for every $z \in \overset{\times}{D}(0;r)$. Show that $\lim_{z \to 0} f(z) g(z) = 0$.

11. Let $\{z_n\}_{n=1}^{\infty}$ be a convergent sequence with $\lim_{n \to \infty} z_n = l \neq 0$. Show that there is a choice of argument such that $\{\text{"arg"}(z_n)\}_{n=1}^{\infty}$ is convergent. Considering the example, $\left\{ z_n = \frac{1}{n} e^{in} \right\}$, show that the hypothesis $l \neq 0$ cannot be dropped

12. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a continuous function. Assume that for every $\alpha \in \partial\Omega$ (boundary of Ω), $\lim_{z \to \alpha} f(z)$ exists.
Define $\tilde{f} : \overline{\Omega} \to \mathbb{C}$ by $\tilde{f}(z) = \begin{cases} f(z) \, ; & \text{if } z \in \Omega \\ \lim_{z \to \alpha} f(z) \, ; & \text{if } z \in \partial\Omega \end{cases}$.
Show that $\tilde{f}$ is continuous.

13. Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$. Assume that for every open set $\mathcal{U} \subset \mathbb{C}$, there exists an open set $V \subset \mathbb{C}$ such that $f^{-1}(\mathcal{U}) = V \cap A$.
Show that f is continuous

14. Let $K \subset \mathbb{C}$ be a compact set and $f : K \to \mathbb{C}$ a continuous function. Use the "sequential compactness" property to show that $f(K)$ is compact.

15. Show that composition of continuous functions is continuous (use Ex. 13).

16. Let $A \subset \mathbb{C}$ be a closed set and $z_0 \in \mathbb{C}$. Define
$\operatorname{dist}(z_0; A) := \inf \{ |z - z_0| : z \in A \}$. Show that $\operatorname{dist}(z_0; A) = 0 \iff z_0 \in A$.

17. Let $S = \{ (a,b,c) \in \mathbb{R}^3 : a^2+b^2+c^2 = 1 \}$ be the unit sphere (centered at $(0,0,0)$).
Let $\rho : S \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection. Prove that
$(N = (0,0,1))$ (i) If C is a circle lying on S, then $\rho(C)$ is either a straight line (if C passes through N); or a circle.
(ii) For $P = (a,b,c)$, let $P^* = (a,b,-c)$ (reflection in XY plane).
Show that $\rho(P) = z \Rightarrow \rho(P^*) = \dfrac{z}{|z|^2}$ (compare with Problem 23 of Set 1).

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18. Let $\Omega \subset \mathbb{C}$ be open. Let $A = \mathbb{C} \setminus \Omega$. For each $n \geq 1$, define
$K_n := \overline{D}(0;n) \cap \{ z \in \mathbb{C} : \operatorname{dist}(z; A) \geq \frac{1}{n} \}$. Show that

(i) $K_n \subset K_{n+1}$ for each n.
(ii) K_n is compact
(iii) for every $\alpha \in \Omega$, there exists $n \geq 1$ s.t. $\alpha \in K_n$

"open sets can be approximated by compact sets".