Problem Set 3

[Holomorphic = $\mathbb{C}$-differentiable.
Contours are assumed counterclockwise]

1. In each of the following cases, check the Cauchy-Riemann equations. If these hold, express the given function in terms of $z = x+yi$ variable.

   (a) $(2+y) + xi$; (b) $(x^3 - 3xy^2) + (3x^2y - y^3)\,i$

   (c) $\dfrac{x^2-y^2}{(x^2+y^2)^2} - \dfrac{2xy}{(x^2+y^2)^2}\,i$; (d) $(e^x + e^{-x})(\cos(y) + \sin(y)\,i)$.

2. Express Cauchy-Riemann equations in polar coordinates (r,θ).

3. Prove that Laplace equation $\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = 0$ in polar coordinates takes the following form: $r^2 \dfrac{\partial^2 g}{\partial r^2} + r\dfrac{\partial g}{\partial r} + \dfrac{\partial^2 g}{\partial \theta^2} = 0$. Use this to show that if g is harmonic and independent of θ, then $g = A\ln(r) + B$ (for constants $A, B \in \mathbb{R}$).

4. Let $f(x+yi) = u(x,y) + iv(x,y)$ be a holomorphic function. Assume $z_0 \in \mathbb{C}$ is such that $f'(z_0) \neq 0$. Show that the level curves ($z_0 = x_0 + y_0 i$)

   $$u(x,y) = u(x_0,y_0) \quad \text{and} \quad v(x,y) = v(x_0,y_0)$$

   meet orthogonally at (x_0,y_0). [Recall: $\langle g_x, g_y\rangle$ is perpendicular to level curve $g(x,y) = C$]

5. Let $n \geq 2$. Describe the level curves $Re(z^n) = 0$; $Im(z^n) = 0$. Explain why do they not meet orthogonally at $z=0$, as in Problem 4.

6. For the following functions - verify Laplace equation. Then compute their harmonic conjugate (i.e. $v(x,y)$ s.t. $u(x,y) + iv(x,y)$ is holomorphic)

   (i) $u(x,y) = \dfrac{x}{x^2+y^2}$ (defined on $\mathbb{C} \setminus \{0\}$)

   (ii) $u(x,y) = x^2 + x - y^2$ $(\Omega = \mathbb{C})$

   (iii) $u(x,y) = e^x(x\cos(y) - y\sin(y))$ $(\Omega = \mathbb{C})$

7. Compute the derivatives of the following functions.

(i) $\dfrac{z}{z-1}$ (ii) $(2z+i)^5$ (iii) e^{z^2}

8. Let γ be any path in $\mathbb{C}$ joining 0 to $1+i$. Compute the following integrals (using primitive of the integrand).

(i) $\displaystyle\int_\gamma \cos(z)\,dz$ (ii) $\displaystyle\int_\gamma z.e^{z^2}dz$ (iii) $\displaystyle\int_\gamma (z^2+1)^2 dz$.

9. Let $n \in \mathbb{Z}$. Verify that $\displaystyle\int_C z^n dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{if } n\neq -1 \end{cases}$. C is any (counterclockwise) contour around 0.

10. Let γ be a contour and S = area of its interior. Show that

$$\int_\gamma \mathrm{Re}(z)dz = i\cdot S \quad ; \quad \int_\gamma \mathrm{Im}(z)\,dz = -S$$ (Hint: use Green's Theorem)

11. Let $\Omega \subset \mathbb{C}$ be an open, connected and simply-connected set, $u:\Omega\to\mathbb{R}$ be a ~~function~~ harmonic function (i.e. $u\in C^2(\Omega)$ and $u_{xx}+u_{yy}=0$). Show that there exists $v:\Omega\to\mathbb{R}$ such that $u+iv:\Omega\to\mathbb{C}$ is holomorphic. (See Lecture 13 - page 8 - for a hint).

12. Prove that $\displaystyle\int_\gamma \frac{1}{z^2+1}dz = 0$ where γ is any contour with $\pm i \in \mathrm{Interior}(\gamma)$

(a) by direct calculation. ; and (b) by the following method:

(i) Let C_R be the circle $|z|=R$, where $R>1$. Show that

$$\int_\gamma \frac{1}{z^2+1}dz = \int_{C_R} \frac{1}{z^2+1}dz$$

(Assume $R>1$ is large enough so that $|z|<R$ for every z on γ)

(ii) Prove the estimate $\left| \int_{C_R} \frac{1}{z^2+1}\, dz \right| \leq \frac{2\pi R}{R^2-1}$.

(iii) Conclude from (i) and (ii) that $\int_\gamma \frac{1}{z^2+1}\, dz = 0$.

13. Compute the following integrals.

(i) $\displaystyle\int_\gamma \frac{z e^z}{(z-a)^3}\, dz$; γ = circle of radius $r>0$ around a $(a \in \mathbb{C})$.

(ii) $\displaystyle\int_\gamma \frac{e^z}{z(1-z)^3}\, dz$; γ = circle of radius $\frac{1}{2}$ around 0.

(iii) $\displaystyle\int_\gamma \frac{z^2+2}{(z-1)(z+i)}\, dz$; γ = circle of radius 2 around 0.

(iv) $\displaystyle\int_\gamma \frac{\sin(z)}{(z-2)^2}\, dz$; γ =