Problem Set 4

1. Let $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_n$ be a polynomial of degree $n \geq 2$; and $z_1, z_2, \dots, z_n \in \mathbb{C}$ be its roots (not necessarily distinct). Let $\zeta \in \mathbb{C}$ be a root of $p'(z)$ and assume that $\zeta \notin \{z_1, \dots, z_n\}$. Show that
$$\boxed{\left(\sum_{j=1}^{n} \frac{1}{|\zeta - z_j|^2} \right) \zeta = \sum_{j=1}^{n} \frac{z_j}{|\zeta - z_j|^2}}$$
(i.e. ζ can be written as $\sum_{j=1}^{n} \lambda_j z_j$; $\lambda_1, \dots, \lambda_n \in (0,1)$; $\lambda_1 + \dots + \lambda_n = 1$. This result is called Gauss-Lucas Theorem. HINT: Compute $\frac{p'(z)}{p(z)}$ from $p(z) = (z-z_1)\cdots(z-z_n)$ and set $z = \zeta$.)

2. Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic function and assume that $f(0) = 0$; $\lim_{z \to \infty} \mathrm{Re}(f(z)) = 0$. Show that $f(z) = 0 \ \forall z \in \mathbb{C}$. (HINT: Show that $e^{f(z)}$ is bounded and use Liouville's Thm.)

3. Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$. Show that $\left| \frac{z - \alpha}{1 - \bar{\alpha} z} \right| < 1 \quad \forall z \in D(0;1)$ (i.e. $|z| < 1$).

4. Consider the following "elementary transformations"
$$\tau_x(z) = z + x \quad ; \quad \sigma_p(z) = pz \quad ; \quad I(z) = \frac{1}{z}. \qquad (x, p \in \mathbb{C}).$$
Let $C(\alpha; r) = \{ z \in \mathbb{C} : |z - \alpha| = r \}$.
circle of radius r, centered at α

(1) Prove that:
$$\tau_x(C(\alpha;r)) = C(\alpha + x; r)$$
$$\sigma_p(C(\alpha;r)) = C(p\alpha; |p| r)$$
$$I(C(\alpha;r)) = \begin{cases} C\left(\frac{\bar{\alpha}}{|\alpha|^2 - r^2}; \frac{r}{\left| |\alpha|^2 - r^2 \right|} \right) & \text{if } |\alpha| \neq r, \\ \text{Line } \{ w : \mathrm{Re}(w\alpha) = \frac{1}{2} \} & \text{if } |\alpha| = r. \end{cases}$$

(2) Determine the image of $C(0;r)$ $(0 < r < 1)$ under the transformation
$$z \longmapsto \frac{z + a}{1 + az} \quad \text{(here } a \in (0,1) \text{ is a fixed real number).}$$

5. Consider the partial fractions of $\dfrac{2z-1}{(z-1)^2(z-4)^3} = \dfrac{A_1}{z-1} + \dfrac{A_2}{(z-1)^2} + \dfrac{B_1}{z-4} + \dfrac{B_2}{(z-4)^2} + \dfrac{B_3}{(z-4)^3}$.

Compute B_2.

6. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of complex numbers. Let

$$S_k = a_1 + a_2 + \cdots + a_k.$$

(i) Show that for every $n, m \in \mathbb{Z}_{\geq 1}$; $n \geq m$, we have

(Abel's transformation) $$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} S_k (b_k - b_{k+1}) - S_{m-1} b_m + S_n b_n$$

(ii) Assume that $\exists\, M \in \mathbb{R}_{>0}$ s.t. $|S_k| < M$ for every k; and $\{b_n\}_{n=1}^{\infty}$ is real, monotonically non-increasing and $\lim_{n\to\infty} b_n = 0$ (i.e. $b_1 \geq b_2 \geq \cdots \to 0$). Prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(Dirichlet's test)

7. Decide whether the following series are convergent or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (iii) $\displaystyle\sum_{n=0}^{\infty} e^{in}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{in\phi}}{n}$ $(0 \leq \phi \leq 2\pi)$ (v) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^2}$ $(0 \leq \theta \leq 2\pi)$ (vi) $\displaystyle\sum_{n=0}^{\infty} \frac{e^{in}}{3^n}$

8. Determine the radii of convergence of the following power series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n z^n}{5^n}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \cos(n)\, z^n$

(iv) $$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{z^n}{n!}$$

($\alpha, \beta, \gamma \in \mathbb{C}$; $\gamma \notin \mathbb{Z}_{\leq 0}$. Hypergeometric Series)

9. Prove that the series $\sum_{n=0}^{\infty} z^{n!}$ diverges for every z such that $z^{\ell}=1$ for some $\ell \in \mathbb{Z}_{\geq 1}$. (Weierstrass).

10. Show that $\dfrac{1}{(1-z)^{n+1}} = \sum_{\ell=0}^{\infty} \binom{n+\ell}{\ell} z^{\ell}$ for every $z \in D(0;1)$, $n \in \mathbb{Z}_{\geq 0}$.

11. Show that if $\sum_{n=0}^{\infty} a_n z^n$ has non-zero radius of convergence then $\sum_{n=0}^{\infty} a_n \dfrac{z^n}{n!}$ has infinite radius of convergence.

12. Use Abel's Theorem : $\boxed{\text{If } \sum_{n=0}^{\infty} a_n \text{ is convergent, then } \lim_{\substack{r\to 1 \\ (0<r<1)}} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} a_n}$ and Problem 7 (iv)

to prove that (i) $\sum_{n=1}^{\infty} \dfrac{\sin(n\phi)}{n} = \dfrac{\pi-\phi}{2}$ $\quad (0<\phi<2\pi)$

(ii) $\sum_{n=1}^{\infty} \dfrac{\cos(n\phi)}{n} = -\log\left(\left|2\sin\left(\frac{\phi}{2}\right)\right|\right)$ $\quad (0<\phi\leq\pi)$

13. Compute the Taylor Series expansion of following functions, around 0.

(i) $\dfrac{z^2}{(z+1)^2}$ (ii) $\log\left(\dfrac{1+z}{1-z}\right)$ (iii) $\sin^2(z)$ (iv) $\dfrac{z}{z^2-4z+13}$

(v) $\dfrac{1}{(z-1)(z-2)}$ (in each case, indicate the radius of convergence).

14. Recall (Lecture 21, §21.3) the definition of Bernoulli numbers :

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z|<2\pi).$$

(i) Show that $B_0=1$, $B_1=-\frac{1}{2}$ and $B_{2k+1}=0$ $\forall\, k\geq 1$.

(ii) Show that $\sum_{\ell=0}^{n} \binom{n+1}{\ell} B_\ell = 0$. (iii) Compute B_2 and B_4.

15. Show that $z \cdot \cot(z) = iz + \dfrac{2iz}{e^{2iz}-1}$ $\left(\cot = \dfrac{\cos}{\sin}\right)$

and use it to write Taylor Series of $z\cot(z)$ in term of Bernoulli numbers (centered at 0)

16. Write the Taylor Series (around 0) of $\log\left(\dfrac{\sin(z)}{z}\right)$ and indicate its radius of convergence. (HINT: take derivative and use Problem 15 above.)

17. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that $A(z)$ is the Taylor Series of a <u>rational function</u> (defined at $z=0$) if, and only if there exist $q_1, q_2, \dots, q_\ell \in \mathbb{C}$ and $N \geq \ell$ s.t.

$$a_n = q_1 a_{n-1} + q_2 a_{n-2} + \dots + q_\ell a_{n-\ell} \quad \forall n \geq N.$$

18. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor Series of a holomorphic function, $(|z| < R)$.

(i) Show that $\forall r < R$, we have $\boxed{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}}$

(ii) For each $r \in [0,R)$, let $M(r) = \operatorname{Max}\{|f(z)| : |z| = r\}$. Prove that $|c_n| \leq \dfrac{M(r)}{r^n}$ $\forall n \geq 0$.

(iii) If there exists $\boxed{\begin{matrix} n \in \mathbb{Z}_{\geq 0} \\ r \in (0,R) \end{matrix}}$ s.t. $|c_n| = \dfrac{M(r)}{r^n}$, then prove that $f(z) = c_n \cdot z^n$.