Problem Set 5

1. Compute the Taylor Series of $\tan^{-1}(z)$ near $z=0$ and its radius of convergence.

2. Same as problem 1 for $\sinh^{-1}(z)$; recall $\sinh(z) = \frac{e^{z} - e^{-z}}{2}$.

3. Compute the first 3 non-zero terms of (i) $\cos(z)^{1/3}$, (ii) $\log(\cos(z))$, (iii) $e^{z/1-z}$.

4. Euler numbers $\{E_{2k}\}_{k=0}^{\infty}$ are defined as coefficients of the Taylor Series expansion of $\sec(z)$ near $z=0$:
$$\sec(z) = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{z^{2n}}{(2n)!}$$
Determine the values of E_0, E_2, E_4. Prove that the following recurrence relation holds:
$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0$$

5. In each of the following cases, solve for $W(z)$ - as a power series in z, with $W(0) = 0$; $W'(0) = 1$. Discuss the radius of convergence of your answer.
   (i) $W'' - z^2 W = 3z^2 - z^4$.
   (ii) $(1-z^2) W'' - 2zW' + n(n+1) W = 0$

6. Let $a, b, c \in \mathbb{C}$; $c \notin \mathbb{Z}_{\leq 0}$. Determine $W(z)$ = a power series in z ; $W(0) = 1$, such that
$$z(1-z) w'' + (c - (a+b+1)z) w' - ab w = 0$$
(hypergeometric equation).
What is its radius of convergence if $a, b \notin \mathbb{Z}_{\leq 0}$.

7. Find the order of vanishing of the following functions at $z=0$

(i) $z^2\left(e^{z^2}-1\right)$ (ii) $6\sin(z^3)+z^3\left(z^6-6\right)$ (iii) $e^{\sin(z)}-e^{\tan(z)}$

8. Does there exist holomorphic function $f: D(0;1)\to\mathbb{C}$ such that

$$f\left(\frac{1}{n}\right)=\begin{cases}0 & \text{if } n \text{ is odd}\\ \frac{1}{n} & \text{if } n \text{ is even}\end{cases} \qquad (n=1,2,3,\dots)?$$

9. Compute the Laurent Series expansions of $f(z)=\dfrac{1}{z(1-z)}$ near $z=0$; $z=1$ and $z=\infty$.

10. Find mistake in the following argument:

$$1+z+z^2+\dots = \frac{1}{1-z}$$

$$z^{-1}+z^{-2}+z^{-3}+\dots = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1} = \frac{-1}{1-z}$$

Adding, we get $\sum_{n=-\infty}^{\infty} z^n = 0$. Coeff of z^0 gives $1=0$.

11. Let $a,b\in\mathbb{C}$; $a\neq b$, and consider $f(z)=\log\left(\frac{z-a}{z-b}\right)$. Show that f has a removable singularity at ∞. Compute its Taylor series expansion near ∞.

12. Discuss the type of singularity (removable / pole / essential) for the indicated value(s).

(i) $\tan(z)$, $z=\frac{\pi}{2}$. (ii) $\dfrac{e^z+1}{e^z-1}$, $z=0$.

(iii) $z\cdot e^{\frac{1}{z}}$; $z=0$ (iv) $\dfrac{z^2+z+1}{z-4}$; $z=4$, $z=\infty$.

13. Prove that $\sin(z)$ has an essential singularity at ∞.

14. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Prove that: if ∞ is a removable singularity of f, then f is constant

15. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that f has a pole of order N at ∞ $(N \in \mathbb{Z}_{\geq 1})$. Prove that f is a polynomial of degree N.

16. Find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ s.t. $f(2k+1) = f(2k+2) = 2k+2$ $\forall\ k = 0, 1, 2, 3, \ldots$

17. Compute the Laurent Series expansion of the following functions near the specified point.

(i) $\dfrac{e^z}{z-1}$; $z = 1$

(ii) $\dfrac{1}{z(z-1)}$ near $z = 2$.

(iii) $\operatorname{cosec}(z)$; $z = 0$

(iv) $\cot(z) - \dfrac{1}{z}$; $z = 0$

(v) $\dfrac{1}{\sin(z) - \sin(a)}$; $z = a$

(vi) $\dfrac{z^2+1}{z+2}$ near $z = \infty$.