Problem Set 6

1. Compute the following residues : (i) $\operatorname{Res}_{z=0} \dfrac{1}{z^3 - z^5}$. (ii) $\operatorname{Res}_{z=0} e^{1/z}$

   (iii) $\operatorname{Res}_{z=\infty} e^{z}$ (iv) $\operatorname{Res}_{z=0} \dfrac{1}{z(1-e^{-z})}$ (v) $\operatorname{Res}_{z=3i} \dfrac{e^z}{z^2(z^2+9)}$

   (vi) $\operatorname{Res}_{z=n\pi} \operatorname{cosec}(z)$ $(n \in \mathbb{Z})$ (vii) $\operatorname{Res}_{z=\infty} \dfrac{z^2+z+1}{z^2(z-1)}$.

2. Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function with finitely many singularities $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Show that $\left(\sum_{j=1}^{n} \operatorname{Res}_{\alpha_j} (f(z)) \right) + \operatorname{Res}_{\infty}(f(z)) = 0$

3. Assume that $f(z) = c_0 + \dfrac{c_{+1}}{z} + \dfrac{c_2}{z^2} + \cdots$ $\forall\, z$ s.t. $|z| > R$. $(R \in \mathbb{R}_{>0})$.

   Compute $\operatorname{Res}_{\infty} (f(z)^2)$.

4. Let $\alpha \in \mathbb{C}$ and assume that $\varphi : D(\alpha; r) \to \mathbb{C}$ is holomorphic $(r \in \mathbb{R}_{>0})$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function s.t. $\alpha \notin \Omega$; $\overset{\times}{D}(\alpha; r) \subset \Omega$ and $\operatorname{Res}_{z=\alpha} (f(z)) = A$, α is a simple pole of f.

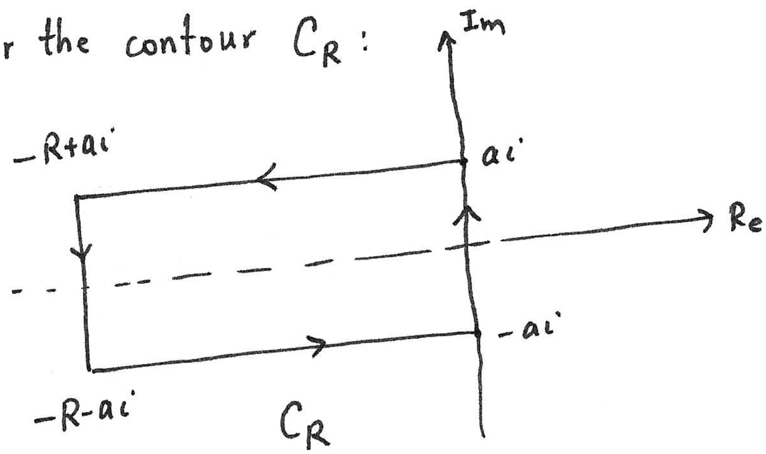
   Compute $\operatorname{Res}_{z=\alpha} (\varphi(z) f(z))$.

5. Let $f : \Omega \dashrightarrow \mathbb{C}$ be a meromorphic function ; $\varphi : \Omega \to \mathbb{C}$ a holomorphic function. Let $\alpha \in \Omega$ be a pole (of order N) of f. $(N \in \mathbb{Z}_{\geq 1})$

   Compute $\operatorname{Res}_{z=\alpha} \left(\varphi(z) \dfrac{f'(z)}{f(z)} \right)$.

6. Same as 5 , except α is a zero of f ; of order $N \in \mathbb{Z}_{\geq 1}$.

7. Let $a, R \in \mathbb{R}_{>0}$ and consider the contour C_R:



Compute

$$\frac{1}{2\pi i} \int_{C_R} \frac{e^z}{\cos(z)}\, dz.$$

Does it have a limit as $R \to \infty$?

8. Compute the following integrals:

(i) $$\int_0^{2\pi} \frac{d\theta}{(a+b\cos(\theta))^2} \qquad (a>b>0) \text{ real numbers}$$

(ii) $$\int_0^{2\pi} \frac{d\theta}{1-2p\cos(\theta)+p^2} \qquad (p \in \mathbb{C},\ p \neq \pm 1)$$

(iii) $$\int_0^{\pi} \tan(x+ia)\, dx \qquad (a \in \mathbb{R})$$

(iv) $$\int_0^{2\pi} \cot(x+a)\, dx \qquad (a \in \mathbb{C},\ \mathrm{Im}(a)>0)$$

(v) $$\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin(\theta))\, d\theta \qquad (n \in \mathbb{Z}) \quad \text{(a bit challenging)}$$

(Hint: compute $$\int_0^{2\pi} e^{\cos(\theta)} \left(\cos(n\theta - \sin(\theta)) + i \sin(n\theta - \sin(\theta))\right) d\theta$$
$$= \int_0^{2\pi} e^{\cos(\theta)} e^{i(n\theta - \sin(\theta))}\, d\theta$$ and take the real part.)

9. Compute $\text{P.V.} \int_{-\infty}^{\infty} Q(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} Q(x)\,dx$ in the following cases.

(i) $Q(x) = \dfrac{x}{(x^2+4x+13)^2}$ (ii) $Q(x) = \dfrac{x^2}{(x^2+a^2)^2}$ $(a \in \mathbb{R}_{>0})$.

(iii) $Q(x) = \dfrac{1}{(x^2+1)^n}$ $(n \in \mathbb{Z}_{\geq 1})$ (iv) $Q(x) = \dfrac{1}{x^{2n}+1}$ $n \in \mathbb{Z}_{\geq 1}$

10. Use Jordan's Lemma to evaluate the following infinite integrals (principal value).

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2-2x+10}\,dx$ and $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2-2x+10}\,dx$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2}\,dx$ $(a, b \in \mathbb{R}_{>0})$.

11. For the following problems, use indented contours to compute the (principal value) of given integrals.

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\,dx$ $(t \in \mathbb{R})$. (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2-5x+6}\,dx$

(iii) $\displaystyle\int_{0}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx$ $(a, b \in \mathbb{R}_{>0})$ (iv) $\displaystyle\int_{0}^{\infty} \frac{\sin^2(x)}{x^2}\,dx$ $\left(\text{Hint: consider integrating } \dfrac{e^{2iz}-1}{z^2} \text{ over } C_{r,R}\right)$

(v) $\displaystyle\int_{0}^{\infty} \frac{\sin^3(x)}{x^3}\,dx$ $\left(\text{HINT: use the integral } \displaystyle\int_{C_{r,R}} \frac{e^{3iz}-3e^{iz}+2}{z^3}\,dz\right)$