Problem Set 7

In Problems 1-5, use Rouché's Theorem (Lecture 30).

1. Determine number of solutions of the following equations that lie in $D(0;1) = \{z \in \mathbb{C} : |z| < 1\}$.

   (a) $z^9 - 2z^6 + z^2 - 8z - 2 = 0$; (b) $2z^5 - z^3 + 3z^2 - z + 8 = 0$

   (c) $z^7 - 5z^4 + z^2 - 2 = 0$.

2. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$. Assume that $\exists$ k ~~st~~. and a contour C s.t. $|a_k z^k| > \left| \sum_{\substack{j=0 \\ (j \neq k)}}^{n} a_j z^j \right|$ $\forall z$ on C.

   Prove that $P(z) = 0$ has k solutions within C ~~and~~ if $0 \in \text{Interior}(C)$;
   has 0 solutions within C if $0 \notin \text{Interior}(C)$.

3. Find number of solutions of $z^4 - 5z + 1 = 0$ that lie in $D(0;1) = \{z \mid |z| < 1\}$ and that lie in $\text{Ann}(0;1,2) = \{z \mid 1 < |z| < 2\}$.

4. Find number of solutions of $z^4 - 8z + 10 = 0$ that lie in $\text{Ann}(0;1,3)$ $(\{z \mid 1 < |z| < 3\})$.

5. Let $\lambda \in \mathbb{R}$, $\lambda > 1$. Prove that the equation $\lambda = z + e^{-z}$ has exactly one solution (real, positive) in $\{z \mid \text{Re}(z) > 0\}$.

6. Prove that $\sin(\phi) = 2^k \sin\left(\frac{\phi}{2^k}\right) \cdot \prod_{j=1}^{k} \cos\left(\frac{\phi}{2^j}\right)$ $\forall k \in \mathbb{Z}_{\geq 0}$.

   Use it to prove that $\frac{\sin(\alpha)}{\alpha} = \prod_{n=1}^{\infty} \cos\left(\frac{\alpha}{2^n}\right)$ and

   $$\frac{\pi}{2} = \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots$$

7. Show that $\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right)$. (Wallis' formula).

(Hint: use the product formula $\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2} \right)$.)

8. Prove the following identities: (see Mittag-Leffler and Weierstrass Theorems - Lectures 32, 33).

(i) $$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 + 4n^2\pi^2}$$

(ii) $$\operatorname{cosec}^2(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n\pi)^2}$$ (Hint: $\frac{d}{dz} \cot(z) = -\operatorname{cosec}^2(z)$)

(iii) $$\cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

(iv) $$\frac{e^z - 1}{z} = e^{z/2} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2\pi^2} \right)$$

(v) $$\frac{e^{az} - e^{bz}}{(a-b)\, z} = e^{\frac{1}{2}(a+b)z} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{(a-b)^2 z^2}{4n^2\pi^2} \right)$$

In the remainder of the problems: $\Gamma(z)$ denotes Euler's gamma function

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \right) : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}.$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right)$$

here $\gamma = \lim_{N \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{N} - \ln(N) \right)$. (Euler-Mascheroni constant)

9. Show that $\int_0^\infty \ln(t) \cdot e^{-t}\, dt = -\gamma$. (Hint: See Lecture 35, § 35·6)

10. Show that $\psi(z+1) = \psi(z) + \frac{1}{z}$. and

$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z).$$

[recall: from above: $\psi = \Gamma'/\Gamma$]

11. Prove that, $\forall\, n \in \mathbb{Z}_{\geq 0}$; $\psi^{(n)}(z)$ solves the following equation

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + \frac{(-1)^n \cdot n!}{z^{n+1}}.$$

Use this to write a solution of $F(z+1) = F(z) + \frac{1}{z(z-1)^3}$.

12. Use gamma function to solve : $F(z+1) = F(z) \cdot \left(\frac{z^2-2}{z(z+1)}\right)$.

13. For $y \in \mathbb{R}$, $(y \neq 0)$ show that $|\Gamma(iy)| = \sqrt{\dfrac{2\pi}{y(e^{\pi y} - e^{-\pi y})}}$.

[Hint: use $\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$,

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$]

14.* Let $T = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x ;\ 0 \leq y ;\ x+y \leq 1\}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous functions. Show that, $\forall\, a, b \in \mathbb{R}_{>0}$:

$$\iint_T f(x+y)\, x^{a-1} y^{b-1}\, dx\, dy = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \int_0^1 f(t)\, t^{a+b-1}\, dt$$

(Hint : change of variables $t = x+y$; $s = \frac{x}{x+y}$.

$x = st$; $y = t(1-s)$)

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* Dirichlet (1829)