Problem Set 8

1. For $l \in \mathbb{Z}_{\geq 2}$, let $\zeta(l) = \sum_{n=1}^{\infty} \frac{1}{n^l}$. Show that the Taylor series expansion of $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ near $z = 1$ is given by

$$\psi(z) = -\gamma + \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(m+1)(z-1)^m \qquad (|z-1| < 1)$$

Here $\gamma = \lim_{N\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{N} - \ln(N)\right)$ is the Euler-Mascheroni constant.

2. Use $\gamma = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{e^{-t}}{t}\right) dt$ and $\psi'(z) = \int_0^{\infty} \frac{t}{1-e^{-t}} e^{-zt}\, dt$ to prove

$$\psi(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}}\right) dt \quad \left(\text{Gauss' formula for } \psi = \frac{\Gamma'}{\Gamma}\right).$$

3. Prove that $\psi(1) - \psi(\frac{1}{2}) = 2 . \ln(2)$ using Gauss' formula (Problem 2).

4. Prove that (i) $\gamma = \int_0^1 \frac{1-e^{-t}}{t}\, dt - \int_1^{\infty} \frac{e^{-t}}{t}\, dt$ (Gauss)

(ii) $\gamma = \int_0^{\infty} \left(\frac{1}{t(1+t)} - \frac{e^{-t}}{t}\right) dt$ (Dirichlet)

(Hint for (i): $1 + \frac{1}{2} + \dots + \frac{1}{n} = \int_0^1 \frac{1-(1-t)^n}{t}\, dt$.

Show that $\gamma = \lim_{N\to\infty} \left(\int_0^1 \left(1 - \left(1 - \frac{t}{N}\right)^N\right) \frac{dt}{t} - \int_1^N \left(1 - \frac{t}{N}\right)^N \frac{dt}{t}\right)$

and use $e^{-t} = \lim_{N\to\infty} \left(1 - \frac{t}{N}\right)^N$.)

5. Given $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$, the Borel sum $\sum_{n=0}^{\infty} a_n$ is defined as

$$\sum_{n=0}^{\infty} a_n \overset{B}{=} \int_0^{\infty} A(t)\, e^{-t}\, dt \quad \text{where } A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

Compute the Borel sum of $1 - 1 + 1 - 1 + \cdots$ (i.e. $a_n = (-1)^n$)

6. Let $a_n = (-1)^n \cdot n!$. Show that the Borel sum

$$\sum_{n=0}^{\infty} (-1)^n n! = e\left(-\gamma + 1 - \frac{1}{2\cdot 2!} + \frac{1}{3\cdot 3!} - \cdots\right) \approx 0.59\ldots$$

(Hint: $\displaystyle\int_0^{\infty} \frac{e^{-t}}{1+t}\, dt = e \cdot \int_1^{\infty} \frac{e^{-u}}{u}\, du = e \cdot \left(\int_1^{\infty} \frac{e^{-u}}{u}\, du - \int_0^1 \frac{1-e^{-u}}{u}\, du + \int_0^1 \frac{1-e^{-u}}{u}\, du \right)$

$\displaystyle = e\left(-\gamma + \int_0^1 \frac{1-e^{-u}}{u}\, du\right)$

by Problem 4 (i).)

7. Recall that the Laplace transform $\mathcal{L}\phi(z)$ for $\phi : \mathbb{R}_{>0} \to \mathbb{C}$ is defined as $\mathcal{L}\phi(z) = \displaystyle\int_0^{\infty} \phi(t)\, e^{-zt}\, dt$. Compute $\mathcal{L}\phi(z)$ in the following cases - and indicate the right-half plane where your answer is valid.

(i)* $\phi(t) = \dfrac{1}{1+t}$ $\quad$ ~~(ii) $\phi(t) = (t-a)^n$~~ $\quad$ (ii) $\phi(t) = t^n$ $(n \in \mathbb{Z}_{\geq 0})$

(iii) $\phi(t) = e^{\kappa t}$ $\quad$ $(\kappa \in \mathbb{R})$

8. Let $\phi(t) = n$ if $t \in [n, n+1)$ $\forall\, n \in \mathbb{Z}_{\geq 0}$.

Compute the Laplace transform of ϕ.

9. Compute the asymptotic expansions of the following Laplace transforms, using Watson's Lemma

(i) $\displaystyle\int_0^\infty \frac{e^{-tz}}{1+t}\,dt$ (ii) $\displaystyle\int_0^\infty \frac{e^{-tz}}{1+t^2}\,dt$

(iii) $\displaystyle\int_0^\infty \sqrt{1+t}\; e^{-tz}\,dt$.

10. (Inverse Laplace transform). Show that, for $t \in \mathbb{R}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$, $a > 0$, we have

$$\frac{1}{2\pi i}\int_{L_a} e^{zt}\,\frac{dz}{z^{n+1}} = \frac{t^n}{n!}$$

where L_a is the vertical line $\mathrm{Re}(z) = a$ (i.e. $L_a(s) = a + is$; $-\infty < s < \infty$)

11.* Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function with finitely many poles $a_1, \dots, a_n \in \mathbb{C}$. Let $A \in \mathbb{R}$ be such that $\mathrm{Re}(a_j) < A$ for every $1 \leq j \leq n$. Assume that

$$\lim_{\substack{z \to \infty \\ \mathrm{Re}(z) \leq A}} |f(z)| = 0.$$

Show that, for every $t \in \mathbb{R}_{>0}$, we have

$$\int_{L_A} f(z)\, e^{zt}\,dz = 2\pi i \cdot \sum_{j=1}^{n} \operatorname*{Res}_{z=a_j}\left(f(z)\, e^{zt}\right)$$

(here L_A is the vertical line $\mathrm{Re}(z) = A$, traversed from $A - i\infty$ to $A + i\infty$).

12. Use Problem 11 to compute the following integrals. Here (again) L_a is the vertical line $\mathrm{Re}(z) = a$, traversed from bottom to top, and $a \in \mathbb{R}_{>0}$.

(i) $\displaystyle\int_{L_a} \frac{e^{zt}}{z^2+1}\, dz$ (ii) $\displaystyle\int_{L_a} \frac{e^{zt}}{z(z+1)\cdots(z+n)}\, dz$ $(n \in \mathbb{Z}_{\geq 0})$.

13. Show that $\log\left(\frac{z+1}{z}\right)$ is the Laplace transform of $\frac{1-e^{-t}}{t}$.