Problem Set 9

1. Let $f(z) = \dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{C}$; $ad-bc=1$; $c \neq 0$.

   Let C be the circle $\{z \in \mathbb{C} : |cz+d| = 1\}$. Show that f increases areas of regions within C and decreases the area of the regions outside of C.

2. Let $f(z) = \dfrac{z}{z+1}$. Compute the n-fold composition $(\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}})(z)$.

3. Given a Möbius transformation $M \in \mathrm{Aut}(\hat{\mathbb{C}})$, show that there exists a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(A) = 1$ s.t. $M(z) = \dfrac{az+b}{cz+d}$.

   Furthermore, A is unique up to ± 1.

4. Let M be a Möbius transformation. Show that M maps the upper half plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ to itself if, and only if $\exists\, a,b,c,d \in \mathbb{R}$:

   $$M(z) = \frac{az+b}{cz+d} \quad \text{and} \quad ad-bc = 1.$$

5. Let $a,b \in \mathbb{R}$; $a \neq b$. Show that $f(z) = az + bz^{-1}$ maps exterior of the unit disc $\mathbb{D} = \{z : |z| < 1\}$ to exterior of an ellipse.

6. Let $\alpha, \beta \in \mathbb{C}$; $\alpha \neq \beta$. Describe the set of Möbius transformations

   $$\{M \in \mathrm{Aut}(\hat{\mathbb{C}}) : M(\alpha) = \alpha \text{ and } M(\beta) = \beta\}$$

7. Show that $w = \frac{1}{z}$ maps $\Omega = \{z \in \mathbb{C} : |z-1| < 1 \text{ and } |z+i| < 1\}$ to the quadrant $\{\mathrm{Re}(z) > \frac{1}{2} \text{ and } \mathrm{Im}(z) > \frac{1}{2}\}$ bijectively.

8. Compute the following cross-ratios.

(i) $[1 : i : -1 : -i]$ (ii) $[1 : -1 : \infty : 0]$

9. Construct a Möbius transformation M such that:

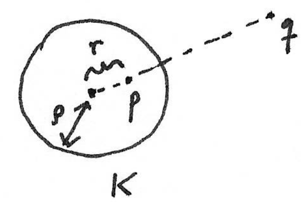
(i) $M(i) = 1$, $M(1) = \infty$; $M(2) = 0$

(ii) M transforms the circle $C(0;1)$ to the line $\{\mathrm{Im}(z) = 2\}$.
(radius 1, center 0)

10. Given 4 distinct points z_1, z_2, z_3, z_4 on $\hat{\mathbb{C}}$, show that

$$[z_1 : z_2 : z_3 : z_4] \in \mathbb{R} \iff z_1, z_2, z_3, z_4 \text{ lie on a circle in } \hat{\mathbb{C}}.$$

11. Let K be a circle in the complex plane, p a point inside K and $q = \rho_K(p)$. (reflection of p through K).



(i) Show that $K = \left\{ z \in \mathbb{C} : \left| \frac{z-p}{z-q} \right| = \frac{r}{\rho} \right\}$

where ρ = radius of K and
r = distance between p and center of K.

(ii) Prove that any circle passing through p and q meets K at right angle

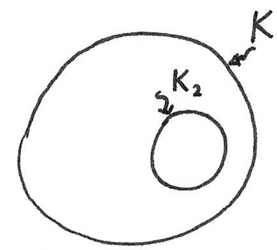
12. Let z_1, z_2, z_3 be three distinct points in $\mathbb{C}$. Show that there is a unique circle K passing through z_1 so that $\rho_K(z_2) = z_3$.

13. (Circles of Apollonius). Let $p \neq q$ be two distinct points in $\mathbb{C}$.
Prove that, for every $\lambda \in \mathbb{R}$, $\lambda \in (0,1)$,

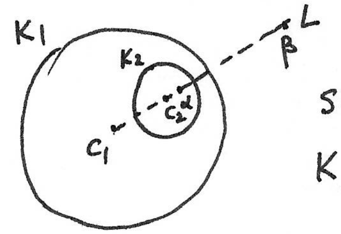
$$K_\lambda = \{ z \in \mathbb{C} : |z-p| = \lambda |z-q| \}$$

defines a circle such that $\rho_{K_\lambda}(p) = q$. What happens at $\lambda = 1$?

14. Let K_1 and K_2 be two circles such that K_2 is in the interior of K_1. Prove that there exists a Möbius transformation M such that $M(K_1)$ and $M(K_2)$ are concentric circles.



[Hint:



show that $\exists\ \alpha, \beta$ on the line joining centers of K_1 and K_2; $\alpha \in$ Interior (K_2); $\beta \in$ Exterior (K_1) such that

$$\beta = \rho_{K_1}(\alpha) = \rho_{K_2}(\alpha).$$

See what happens under the Möbius transformation sending β to ∞.]

15. Let K be a circle in the complex plane; z_1, z_2, z_3 three distinct points on K; $z \in$ Interior (K). Prove that

$$[z_1 ; z_2 : z_3 : \rho_K(z)] = \overline{[z_1 : z_2 : z_3 : z]}.$$

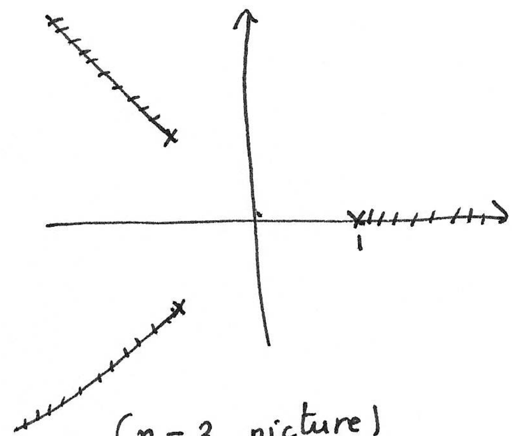
In the following problems, give an <u>explicit</u> conformal equivalence between the given Ω and the upper half plane $\mathbb{H} = \{z : Im(z) > 0\}$.

16. $\Omega = \{z \in \mathbb{C} : -K < Re(z) < K \text{ and } Im(z) > 0\}$. $(K \in \mathbb{R}_{>0})$.

17. $\Omega = \{z \in \mathbb{C} : Re(z) > A ;\ -p < Im(z) < p\}$ $\left(\begin{matrix} A \in \mathbb{R} \\ p \in \mathbb{R}_{>0}\end{matrix}\right)$

18. $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

19. $\Omega = \mathbb{C} \setminus \mathbb{R}_{\geq 1}$. Let $\varphi : \mathbb{H} \to \mathbb{C} \setminus \mathbb{R}_{\geq 1}$ be the conformal equivalence constructed in this problem. Show that $(\varphi(z^n))^{\frac{1}{n}}$ maps $\mathbb{H}$ conformally to $\mathbb{C} \setminus \bigcup_{k=0}^{n-1} \mathbb{R}_{\geq 1} e^{2\pi i \frac{k}{n}}$



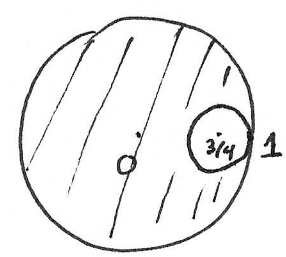
(n = 3 picture)

$\mathbb{C} \setminus \{ t, t e^{\frac{2\pi i}{3}}, t e^{\frac{2\pi i}{3}} : t \in \mathbb{R}_{\geq 1} \}$

20. Let $0 < \phi < \pi$.

$\Omega = \{ z \in \mathbb{C} : |z| < 1 \text{ and } 0 < \arg(z) < \phi \}$.

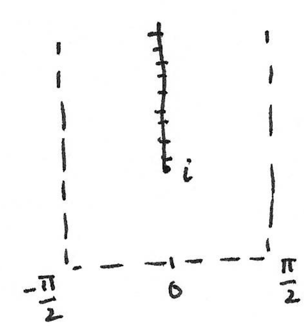
21. Ω = region between tangential circles $\{ |z| = 1 \}$ and $\{ |z - \frac{3}{4}| < \frac{1}{4} \}$



(Problem 21)

22.* $\tilde{\Omega} = \{ z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2} ; \operatorname{Im}(z) > 0 \}$

$\Omega = \tilde{\Omega} \setminus \{ ti : t \in \mathbb{R}_{\geq 1} \}$



23.* $\Omega = \mathbb{H} \setminus \{ ti : 0 < t \leq 1 \}$.