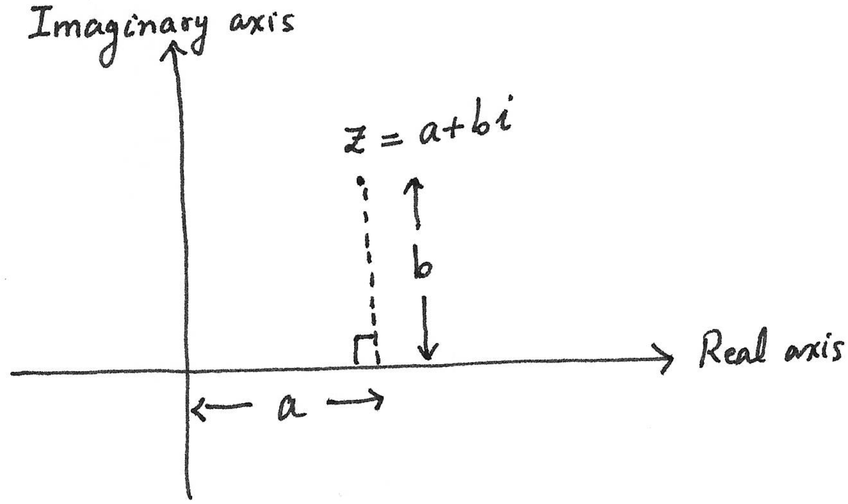
Lecture 0

(0.0) A complex number z is a pair of real numbers (a, b).

Instead of writing it as a pair, we introduce a symbol iota $= i$; and write $z = a + bi$, $a, b \in \mathbb{R}$ = set of real numbers



For $z = a + bi$,

- a is called the real part of z, denoted by $\mathrm{Re}(z)$.
- b is called the imaginary part of z, denoted by $\mathrm{Im}(z)$.

(set of complex numbers is same as two-dimensional real plane.)

(0.1) Addition and multiplication of complex numbers.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ be two complex numbers.

- Addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) i$

  That is, addition is component-wise, exactly as for vectors in $\mathbb{R}^2$.

  In other words: $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$

  $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$

- Multiplaication:

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$$

Remark. – We multiply complex numbers exactly as we multiply two polynomials (in variable i = iota). At the end, we set $i^2 = -1$. In other words, complex multiplication is entirely determined by: (i) multiplication distributes over addition, and (ii) $i^2 = -1$.

Example. Let $z_1 = 1+3i$, $z_2 = 2+i$. Then

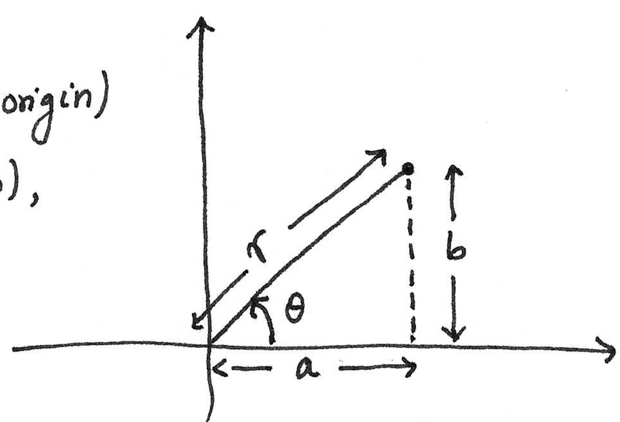
$$z_1 + z_2 = (1+2) + (3+1)i = 3+4i, \text{ and}$$

$$z_1 z_2 = (1+3i)(2+i) = 2 + i + 6i + 3i^2$$

$$= 2-3 + (6+1)i = -1+7i.$$

(0.2) Modulus and argument of a complex number.

Recall that on the two-dimensional real plane, we also have polar coordinates.



For a point (not equal to the origin) with Cartesian coordinates (a,b), polar coordinates are given by

r = distance of the point from the origin

$= \sqrt{a^2+b^2}$

(Cartesian coordinates $(a,b) \neq (0,0)$
Polar coordinates (r,θ))

θ = angle between the positive x-axis and the line joining origin to the given point.

The angle θ is determined by (up to adding an integer multiple of 2π): $\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$ & $\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$.

Conversely, if (r, θ) are polar coordinates of a point, its cartesian coordinates are given by
$$a = r\cos(\theta)$$
$$b = r\sin(\theta)$$

For complex numbers, the polar coordinates are called <u>modulus</u> (or <u>norm</u>) and argument (or phase).

$z = a+bi \neq 0$: Modulus of z $= |z| = \sqrt{a^2+b^2}$ $(\in \mathbb{R}_{>0})$.

Argument of z $= \operatorname{Arg}(z) = \theta$ uniquely determined (upto $\pm 2\pi \cdot$ integer) by
$$\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}} \quad \& \quad \sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$$

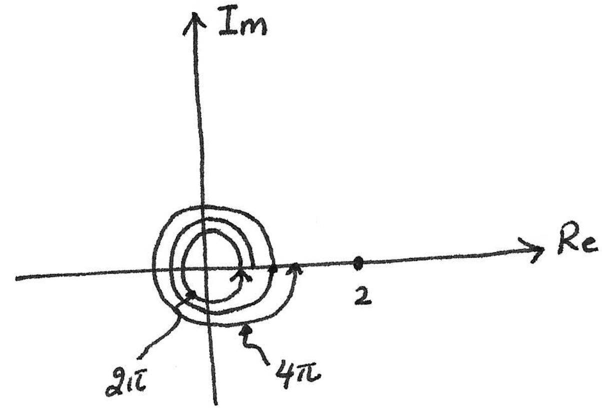
That is,
$$\begin{aligned} z &= a+bi \\ &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\, i\right) \quad (\text{if } a^2+b^2 \neq 0;\ \text{i.e. } z \neq 0). \\ &= |z|\left(\cos(\theta) + \sin(\theta)\, i\right) \end{aligned}$$

Note: $|z| = 0$ if and only if $z = 0$.

$\operatorname{Arg}(z)$ is only defined for $z \neq 0$, and even then there is an ambiguity of adding an integer multiple of 2π.

Example.

$\mathrm{Arg}(2) = 0$, or $\pm 2\pi$, or $\pm 4\pi$, ...



$\mathrm{Arg}(-i) = \frac{-\pi}{2}$, or $\frac{3\pi}{2}$, ...

For definiteness, we require the argument to lie ~~between~~ in $(-\pi, \pi]$

$$-\pi < \arg(z) \leq \pi$$

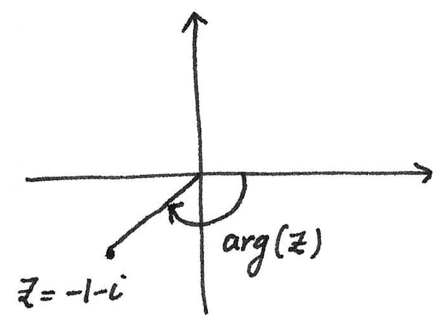
(I will use little 'a' for arg to emphasize this choice)

e.g. $\arg(-1) = \pi$, not $-\pi$.

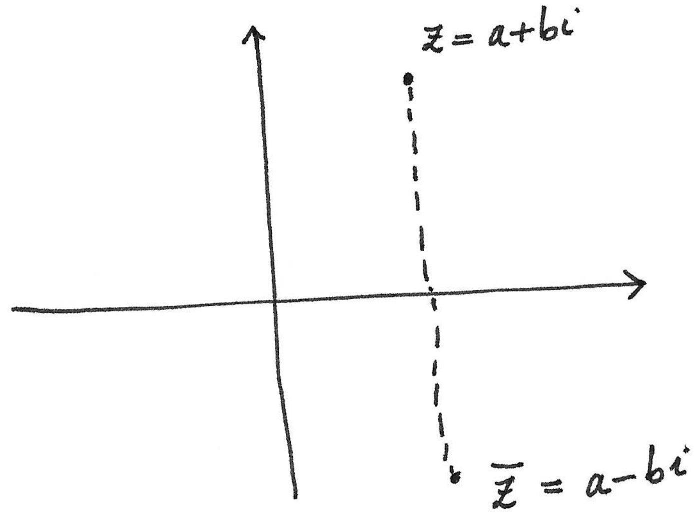
e.g. $z = -1 - i$

$\arg(-1-i) = \frac{-3\pi}{4}$

$\left(\text{not } \frac{5\pi}{4}\right)$.



(0.3) Complex conjugate. For $z = a + bi$, a complex number, we define $\bar{z}$ = <u>conjugate</u> of z = $a - bi$.



Properties.

(i) $\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$,

$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $|z| = |\bar{z}|$ (both equal to $\sqrt{a^2+b^2}$).

(iv) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (Verify this.)

Most importantly, $z\bar{z} = |z|^2$

(Proof : If $z = a+bi$, then $z \cdot \bar{z} = (a+bi)(a-bi)$
$= a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2+b^2 = |z|^2$.)

This means, if $z \neq 0$ (i.e. $|z| \neq 0$) then $z\left(\frac{\bar{z}}{|z|^2}\right) = 1$. That is, every non-zero complex number has a (unique) multiplicative inverse.

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

e.g. Let us compute the real and imaginary parts of $\frac{1+i}{2-3i}$.

We multiply and divide by the conjugate of the denominator :

$$\frac{1+i}{2-3i} = \frac{1+i}{2-3i} \, \frac{2+3i}{2+3i} = \frac{(2-3) + (2+3)i}{2^2 + 3^2}$$

$$= \frac{-1+5i}{13}.$$

$$\operatorname{Re}\left(\frac{1+i}{2-3i}\right) = \frac{-1}{13} \quad \text{and} \quad \operatorname{Im}\left(\frac{1+i}{2-3i}\right) = \frac{5}{13}.$$

(0.4) Notations and summary

- $\mathbb{Z}$ = set of integers = $\{0, \pm 1, \pm 2, \dots\}$
- $\mathbb{Q}$ = set of rational numbers = $\left\{\frac{a}{b} : a, b \in \mathbb{Z}; b \neq 0\right\}$
- $\mathbb{R}$ = set of real numbers.

$\mathbb{C}$ = set of complex numbers (Note: as a set, even as a 2-dimensional real vector space, $\mathbb{C} = \mathbb{R}^2$. The new notation signifies the operation of multiplication).

Properties of $+$ and $\cdot$ on $\mathbb{C}$: Let $z, z_1, z_2, z_3 \in \mathbb{C}$.

- Associativity. $\quad z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$z_1 (z_2 z_3) = (z_1 z_2) z_3$$

- Commutativity: $\quad z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$
- Distributivity: $\quad z(z_1 + z_2) = z z_1 + z z_2$
- Neutral elements (0 & 1): $\quad 0 + z = z$

$$1 \cdot z = z$$

- Inverses exist: $\quad z + (-1) z = 0$

$$\text{if } z \neq 0, \quad z\left(\frac{\bar{z}}{|z|^2}\right) = 1.$$

Side remark. In algebra, any set F admitting binary operations $+$ & $\cdot$ and neutral elements $0, 1 \in F$ satisfying this list of properties is called a <u>field</u>.

Thus, $\quad \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are fields.

$\mathbb{Z}$ is not a field. ($\frac{1}{2} \notin \mathbb{Z}$, for example).

(0.5) Some historical remarks.

(1) Complex numbers owe their existence to the problem of finding roots of a polynomial equation. The formula for the roots of a quadratic equation $\left(ax^2+bx+c=0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \right)$ has been known since around 1800 BC (Babylonians) – and if $b^2-4ac<0$ we were happy with saying "$ax^2+bx+c=0$ has no solutions".

Around 1540 AD, Italian mathematicians* solved the cubic equation

$$x^3 = 3px + 2q \Rightarrow x = \left(q+\sqrt{q^2-p^3}\right)^{\frac{1}{3}} + \left(q-\sqrt{q^2-p^3}\right)^{\frac{1}{3}}$$

Here, we cannot discard the case $q^2-p^3<0$; since a cubic equation always has a real solution (by intermediate value theorem: as $x^3-3px-2q<0$ for $x<<0$ and >0 for $x>>0$; it must be 0 for some $x \in \mathbb{R}$).

e.g. $x^3 = 15x+4$ $(p=5, q=2)$. The formula gives:

$$x = \cancel{2+\sqrt{121}} \left(2+11\sqrt{-1}\right)^{\frac{1}{3}} + \left(2-11\sqrt{-1}\right)^{\frac{1}{3}}.$$

By direct inspection $x=4$ is also a solution.

While Cardona (1545) freely manipulated "impossible numbers", he called them "as subtle as they are useless.

(2) Rafael Bombelli (1526-1572) is often credited for giving the first coherent treatment of the arithmetic of complex numbers.

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* Sciopione del Ferro (1465-1526); Girolamo Cardona (1501-1576); Niccolò Tartaglia (1499-1557)

The "paradox" sketched in the example of $x^3 = 15x + 4$ above is "resolved" by verifying $(2 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-121}$ – hence the formula for roots of a cubic equation works even when $q^2 - p^3 < 0$ – except one has to use algebraic operations on complex numbers.

(3) The term "imaginary" to describe $\sqrt{-d}$ $(d \in \mathbb{R}, d > 0)$ was coined by René Descartes (31/3/1596 – 11/2/1650); around 1620.

(4) The symbol iota $= i$ for $\sqrt{-1}$ was introduced by Johann Carl Friedrich Gauss (30/4/1777 – 23/2/1855). He also popularized the term "complex numbers" that we use now.