Lecture 1

(1.0) Recall: last time we defined complex numbers ($\mathbb{C}$ = set of all complex numbers) and operations of addition, multiplication, taking modulus and argument, and complex conjugation.

We also noted that, as a 2-dimensional real vector space, $\mathbb{C} = \mathbb{R}^2$. As such the familiar notions of vectors in $\mathbb{R}^2$ can be easily translated in the language of complex numbers.

For example, <u>dot product</u> : (in Calculus we use the notation $\langle a,b \rangle$ for vectors in $\mathbb{R}^2$; and define $\langle a,b\rangle \cdot \langle c,d\rangle = ac+bd$).

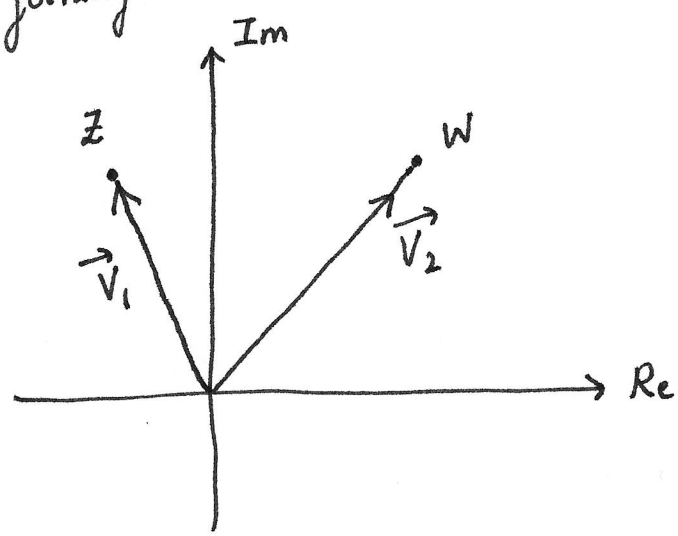
Let $z = a+bi$ and $w = c+di$. Then

$$z\overline{w} = (a+bi)(c-di) = (ac+bd) - (ad-bc)i$$

That is, if $\vec{V_1}$ is the vector joining 0 to z $(\vec{V_1} = \langle a,b\rangle)$
$\vec{V_2}$ is the vector joining 0 to w $(\vec{V_2} = \langle c,d\rangle)$

then

$$\vec{V_1} \cdot \vec{V_2} = \mathrm{Re}(z\overline{w}).$$



Similarly, if we view $\vec{V_1}$ and $\vec{V_2}$ in 3-dim'l vector space

$\vec{V_1} = \langle a,b,0\rangle$, $\vec{V_2} = \langle c,d,0\rangle$, we can take their cross product

$$\vec{V_1} \times \vec{V_2} = \langle 0, 0, \underbrace{ad-bc}_{} \rangle$$

$$= -\mathrm{Im}(z\overline{w}) = \mathrm{Im}(\overline{z}w)$$

Note: for any $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) = \mathrm{Re}(\overline{\alpha})$, $\mathrm{Im}(\alpha) = -\mathrm{Im}(\overline{\alpha})$, translates in this situation to $\vec{V_1} \cdot \vec{V_2} = \vec{V_2} \cdot \vec{V_1}$, $\vec{V_1} \times \vec{V_2} = -\vec{V_2} \times \vec{V_1}$.

(1.1) Geometric meaning of complex multiplication.

The operation of multiplying two complex numbers has a very nice and instructive geometric interpretation - when we switch to polar coordinates. So, let z_1, z_2 be two non-zero complex numbers:

$$r_1 = |z_1|, \quad \theta_1 = \mathrm{Arg}(z_1)$$
$$r_2 = |z_2|, \quad \theta_2 = \mathrm{Arg}(z_2)$$

, i.e.

$$z_1 = r_1\left(\cos(\theta_1) + \sin(\theta_1)i\right) \text{ and } z_2 = r_2\left(\cos(\theta_2) + \sin(\theta_2)i\right)$$

Then, $$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1) + \sin(\theta_1)i\right)\left(\cos(\theta_2) + \sin(\theta_2)i\right)$$
$$= r_1 r_2 \Big(\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right) + \left(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right) i \Big)$$

Using the identities
$$\left\{ \begin{aligned} \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\ \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \end{aligned} \right\}$$

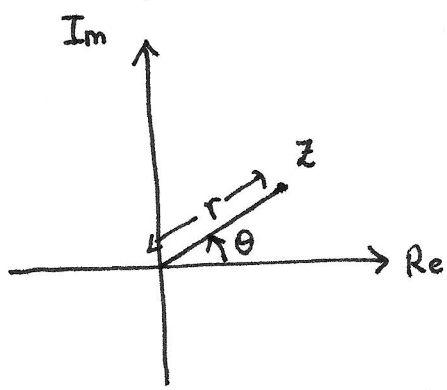
We get $$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\, i \right)$$

That is :

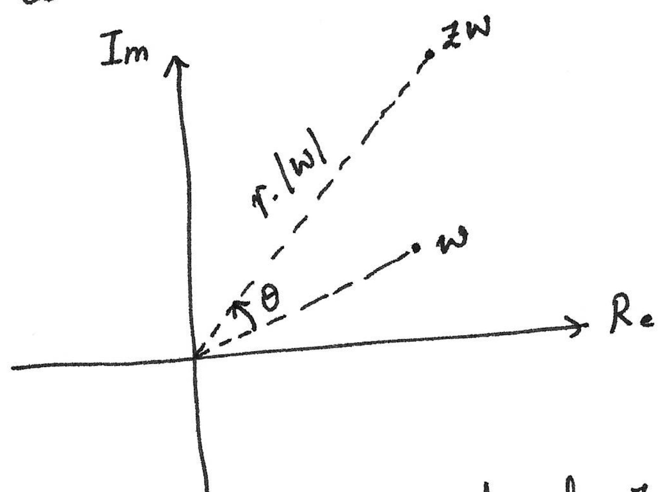
$$\boxed{|z_1 z_2| = |z_1| \cdot |z_2|} \qquad \boxed{\begin{array}{c} \mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \\ (\text{modulo } 2\pi\mathbb{Z}) \end{array}}$$

Geometrically, if $z \in \mathbb{C}$, $z \neq 0$, has modulus r and argument θ,

then multiplication by z =
- scaling by r
- counterclockwise rotation by θ.



$\left(z = r\left(\cos(\theta) + \sin(\theta)\, i\right)\right)$



(Operation of multiplication by z)

(1.2) Application I. de Moivre's formula.

$$\boxed{\left(\cos(\theta) + \sin(\theta)\, i\right)^n = \cos(n\theta) + \sin(n\theta)\, i}$$

for any $n \in \mathbb{Z}$, $n \geq 1$.

Abraham de Moivre (26/5/1667 – 27/11/1754)

- This formula was obtained as an attempt to write $\cos(n\theta)$ & $\sin(n\theta)$ as polynomials in $\cos(\theta)$ and $\sin(\theta)$.

e.g. $n=3$. $\cos(3\theta) + \sin(3\theta)\,i = (\cos(\theta) + \sin(\theta)\,i)^3$

$$= \cos^3(\theta) + 3\cos^2(\theta)(\sin(\theta)\,i) + 3\cos(\theta)(\sin(\theta)\,i)^2 + (\sin(\theta)\,i)^3 \quad \text{(binomial expansion)}$$

$$= \left(\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)\right) + \left(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\right) i$$

$$\Rightarrow \quad \cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$
$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

- This formula is also convenient for computing large powers of a complex number (in some cases).

e.g. Compute $(1+\sqrt{3}\,i)^{12}$.

Sol. $1+\sqrt{3}\,i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) i\right)$

(change to polar form)

$$\Rightarrow (1+\sqrt{3}\,i)^{12} = 2^{12}\left(\cos\left(12\cdot\frac{\pi}{3}\right) + \sin\left(12\cdot\frac{\pi}{3}\right) i\right)$$
$$= 2^{12}\left(\cos(4\pi) + \sin(4\pi)\,i\right)$$
$$= 2^{12} = 4096.$$

(1.3) Application II. n^{th} roots of a complex number ($n \in \mathbb{Z}$; $n \geq 2$).

Example. Let us find all $z \in \mathbb{C}$ for which $z^3 = 1$.

Writing z in polar form $z = r(\cos(\theta) + \sin(\theta)\,i)$, we have

$$z^3 = r^3\left(\cos(3\theta) + \sin(3\theta)\,i\right) = 1.$$

This means $r^3 = 1$ (hence $r = 1$) and

$$3\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$$\theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{6\pi}{3},\ \ldots$$

give same z, as they differ by 2π.

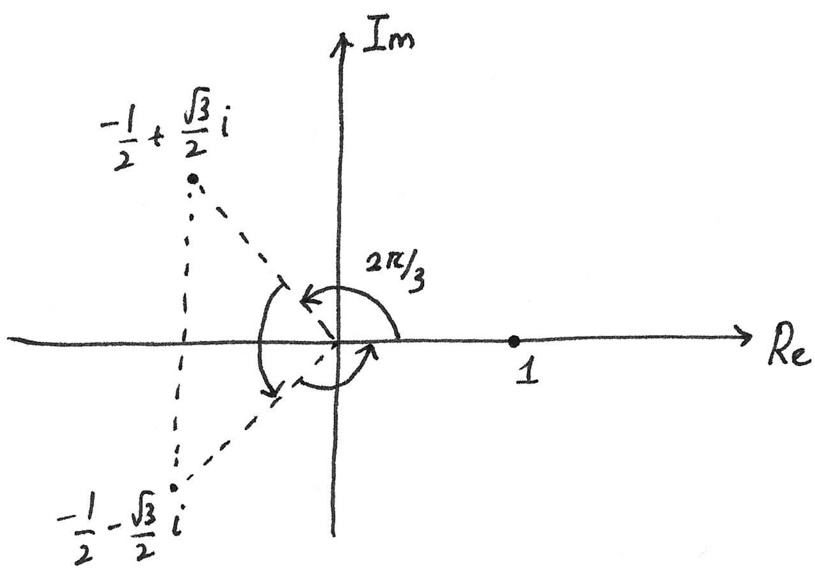
So, $z^3 = 1$ for

$$z = 1 \quad \text{or}$$

$$z = \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)i = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \text{ or}$$

$$z = \cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)i = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

(Three solns of $z^3 = 1$ are obtained by successive rotations of $\frac{2\pi}{3}$.)



In general, we will have to remember that

$$r_1\left(\cos(\theta_1) + \sin(\theta_1)\,i\right) = r_2\left(\cos(\theta_2) + \sin(\theta_2)\,i\right) \iff \begin{array}{l} r_1 = r_2 \\ \theta_1 - \theta_2 \in 2\pi\mathbb{Z} \end{array}$$

Now let $\alpha \in \mathbb{C}$; $\alpha \neq 0$ be fixed, and let us try to solve for $z \in \mathbb{C}$ such that $z^n = \alpha$.

Let $s = |\alpha|$ and $\varphi = \arg(\alpha) \in (-\pi, \pi]$.

Writing $z = r(\cos(\theta) + \sin(\theta) i)$; $z^n = \alpha$ becomes

$$r^n(\cos(n\theta) + \sin(n\theta) i) = s(\cos(\varphi) + \sin(\varphi) i)$$

That is, $r^n = s$ and $n\theta = \varphi, \varphi + 2\pi, \varphi + 4\pi, \dots$

(hence, $r = s^{1/n}$)

$$\Rightarrow \theta = \frac{\varphi}{n}, \frac{\varphi}{n} + \frac{2\pi}{n}, \frac{\varphi}{n} + \frac{4\pi}{n}, \dots, \frac{\varphi}{n} + \frac{(n-1)2\pi}{n}, \frac{\varphi}{n} + \frac{n \cdot 2\pi}{n}$$

give same z.

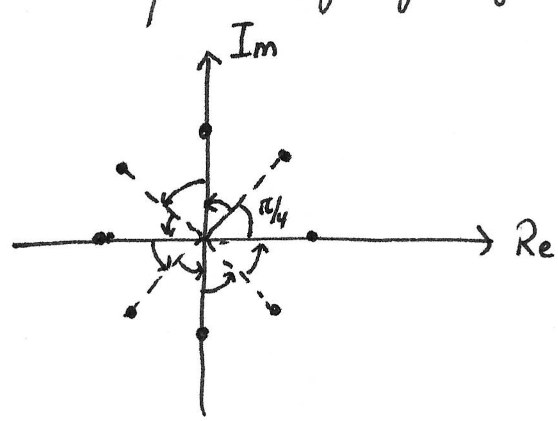
Thus, $z^n = \alpha$ is solved by $z \in \mathbb{C}$ such that $|z| = |\alpha|^{1/n}$ and

$$\theta = \frac{\varphi}{n} + \frac{k \cdot 2\pi}{n}, \quad 0 \leq k \leq n-1 \quad (n \text{ solutions})$$

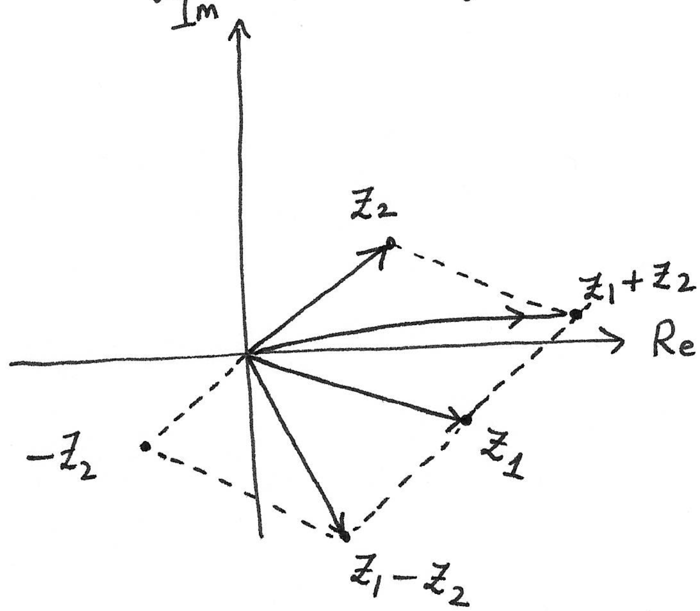
e.g. $z^3 = -8$. Start from one solution, say -2, and rotate by $\frac{2\pi}{3}$.

$z^8 = 1$. Start from 1 and keep rotating by $\frac{2\pi}{8} = \frac{\pi}{4}$

8 solutions of: $z^8 = 1$



(1.4) Geometric meaning of addition of complex numbers is the same as the one for vectors – triangle or parallelogram law



Triangle inequality.

(i) $|z_1+z_2| \leqslant |z_1| + |z_2|$

Proof.

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \qquad (\text{for any } \alpha \in \mathbb{C},\ |\alpha|^2 = \alpha \cdot \bar{\alpha})$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \qquad \left(\begin{array}{c}\text{for any } \alpha,\beta \in \mathbb{C} \\ \overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta} \text{ and} \\ \bar{\bar{\alpha}} = \alpha\end{array}\right)$$

$$= |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2})$$

$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) \qquad (\alpha + \bar{\alpha} = 2\,\mathrm{Re}(\alpha))$$

Now, for any complex number α, $\mathrm{Re}(\alpha) \leqslant |\alpha|$ (because, for $x \in \mathbb{R}$, $x \leqslant |x| = \sqrt{x^2} \leqslant \sqrt{x^2+y^2}$). Hence, we get

$$|z_1+z_2|^2 \leqslant |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \qquad \left(\begin{array}{c}\text{recall: } |\alpha\beta| = |\alpha|\cdot|\beta| \\ \text{and } |\bar{\alpha}| = |\alpha|\end{array}\right)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$= (|z_1| + |z_2|)^2$$

$$\Rightarrow \quad |z_1+z_2|^2 \leqslant |z_1| + |z_2|. \qquad \square$$

Triangle inequality (ii) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

Proof. $|z_1| = |(z_1 - z_2) + z_2|$

$\leq |z_1 - z_2| + |z_2|$ (by the previous triangle inequality)

$\Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|.$

Switching the roles of z_1 and z_2, we get

$|z_2 - z_1| \geq |z_2| - |z_1|$. Since $|z_1 - z_2| = |z_2 - z_1|$

we get $|z_1 - z_2| \geq \text{Max}\,(|z_1| - |z_2|,\ |z_2| - |z_1|)$

$= \big||z_1| - |z_2|\big|$

$\square$