Lecture 2

(2.0) The aim of this lecture is to define exponential, sine and cosine of a complex number. All these definitions are based on the following formula due to Euler*:

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

(2.1) The number $e = 2.71828\ldots$

- Jacob Bernoulli (6/1/1655 – 16/8/1705) was the first mathematician to study the following limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

and observe that the limit exists, in the context of a problem of compound interest. Incidently, this number was denoted by 'b' (for Bernoulli) in a correspondence between Leibniz and Huygens around 1690.

- Euler came across the same number while studying the problem of "population growth".

$$\begin{cases} f'(t) = k f(t) & \text{[rate of growth is proportional to current population]} \\ f(0) = c & \text{[initial population]} \end{cases}$$

For $k = c = 1$, he solved for f via series expansion:

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* Leonhard Euler 15/4/1707 – 18/9/1783

$$f(t) = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

where $\quad e = f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.71828\ldots$

Remark. – • This formula for e is much better than $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, computationally speaking. For instance, if we want accuracy up to 4 decimal places, we can obtain it for $n=7$ in $1+\frac{1}{2!}+\ldots+\frac{1}{7!}$, and for $n=10^5$ in $(1+10^{-5})^{10^5}$.

- It is likely that Euler used 'e' for population "<u>e</u>xplosion".
- The general problem $f'(t) = k f(t)$, $f(0) = c$, is solved by $c \cdot e^{kt}$.

(2.2) Properties of e^t: Euler derived properties of e^t from the differential equation defining it:

$$\begin{cases} e^t \text{ is the unique function satisfying} \\ \frac{d}{dt}(e^t) = e^t \quad \text{and} \quad e^0 = 1. \end{cases}$$

$\underline{e^{t_1+t_2} = e^{t_1} \cdot e^{t_2}}$. Viewing both sides as functions of t_1 (keeping t_2 fixed) we see that both sides solve:

$$f'(t_1) = f(t_1)$$
$$f(t_1=0) = e^{t_2}$$

So they must be equal.

Second proof (using series expansion and binomial theorem):

$$e^{t_1+t_2} = \sum_{n=0}^{\infty} \frac{(t_1+t_2)^n}{n!}$$

$\left(\text{Notation } \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N \right.$

$\left. \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \right)$

Recall the binomial formula

$$(A+B)^n = A^n + n A^{n-1} B + \binom{n}{2} A^{n-2} B^2 + \dots + n A B^{n-1} + B^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k \ ; \text{ where } \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Hence, we get : $$e^{t_1+t_2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} t_1^{n-k} t_2^k \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{k,l \geq 0 \\ k+l=n}} \frac{t_1^l}{l!} \frac{t_2^k}{k!} \right) = \left(\sum_{l=0}^{\infty} \frac{t_1^l}{l!} \right) \left(\sum_{k=0}^{\infty} \frac{t_2^k}{k!} \right)$$

$$= e^{t_1} \cdot e^{t_2}$$

(We will return to algebraic operations with infinite series later in the course – don't worry.)

(2.3) Euler's formula* $\boxed{e^{it} = \cos(t) + \sin(t)\, i}$

Euler's proof.- Consider the differential equation $f''(t) = -f(t)$.

[This equation appears in the description of oscillating object - eg. attached to a spring ; or pendulum with small angle.]

We know two solutions of this equation : $\cos(t)$ and $\sin(t)$.

In general, if $f(0) = a$, $f'(0) = b$ (initial conditions) then $f(t) = a\cos(t) + b\sin(t)$.

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* Feynman (Lectures on Physics - vol I, Ch. 22) calls it "the most remarkable formula in mathematics"

Now $\frac{d^2}{dt^2}\left(e^{kt}\right) = k^2 e^{kt}$. Armed with $\sqrt{-1}$, we can get e^{it} as another solution of $f'' = -f$. The equation is 2^{nd} order-so it cannot have 3 independent solutions. Working out the initial conditions

$$e^{it}\Big|_{t=0} = 1 \qquad \frac{d}{dt}\left(e^{it}\right)\Big|_{t=0} = i$$

we get $e^{it} = \cos(t) + i\sin(t)$.

$\square$

Proof with Taylor Series. - Recall the Taylor series of cosine & sine:

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \qquad \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

Setting $x = it$ in $e^x = 1 + x + \frac{x^2}{2!} + \dots$, we get

$$e^{it} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \dots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} \dots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)$$

$$= \cos(t) + i\sin(t)$$

$\square$

Examples. - $e^{\pm 2\pi i} = e^{\pm 4\pi i} = \dots = 1$.

$e^{\pi i} = -1$. $\qquad e^{\frac{\pi}{2} i} = i$.

(2.4) From now onwards, we will write $e^{i\theta}$ for $\cos(\theta) + \sin(\theta) i$.

That is, for a non-zero complex number α with $|\alpha| = r$, $\arg(\alpha) = \theta$;

$$\alpha = r\left(\cos(\theta) + \sin(\theta) i\right) = r \cdot e^{i\theta}$$

- de Moivre's formula: $\quad (e^{i\theta})^n = e^{in\theta}$
- complex conjugation: $\quad \overline{r e^{i\theta}} = r e^{-i\theta}$
- inverse $\quad \dfrac{1}{r \cdot e^{i\theta}} = \dfrac{1}{r} \cdot e^{-i\theta}$

much easier to remember

(2.5) e^z for $z = x + yi \in \mathbb{C}$ is now defined as:

$$e^z = e^x \cdot e^{yi} \quad \left(= e^x \left(\cos(y) + \sin(y) i\right)\right)$$

That is e^z is always a non-zero complex number with

$$|e^z| = e^{\mathrm{Re}(z)} \quad (\text{positive real number})$$

$$\arg(e^z) = \mathrm{Im}(z) \quad (\text{modulo } 2\pi\mathbb{Z})$$

Check: $\quad e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$

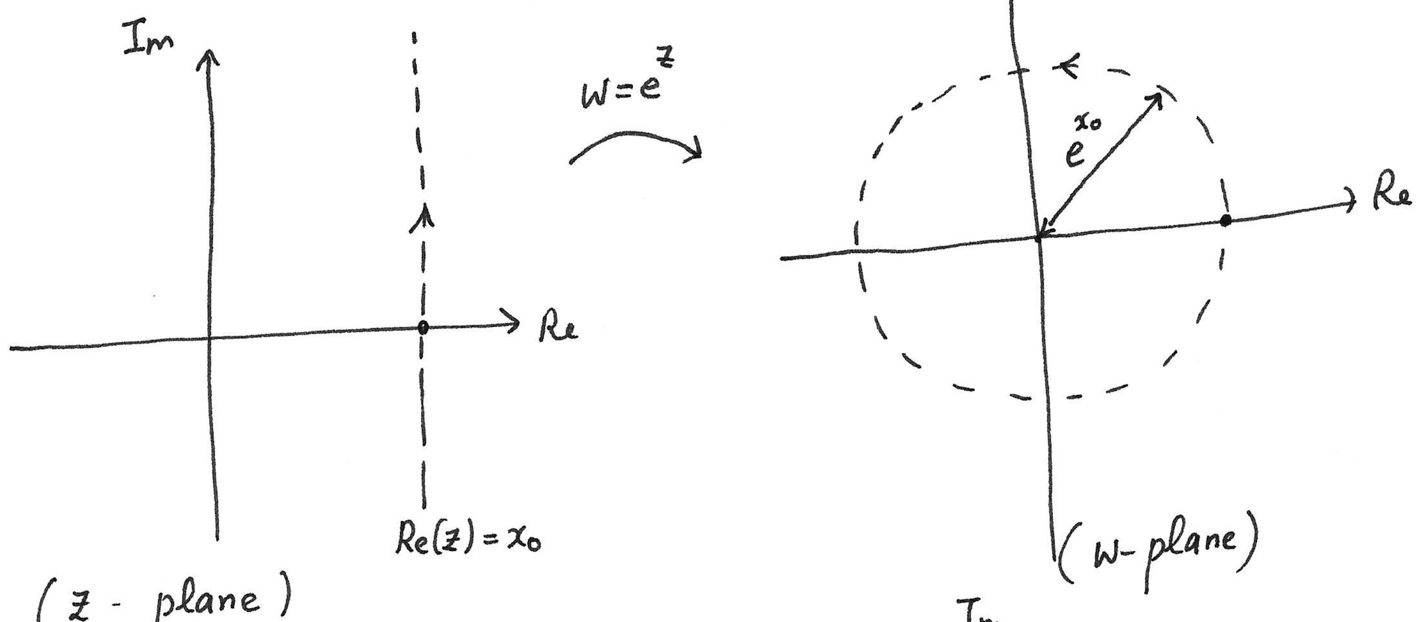
$e^z = 1$ if and only if $z \in 2\pi i \mathbb{Z}$ (i.e. $z = 2\pi i k$ where $k \in \mathbb{Z}$)

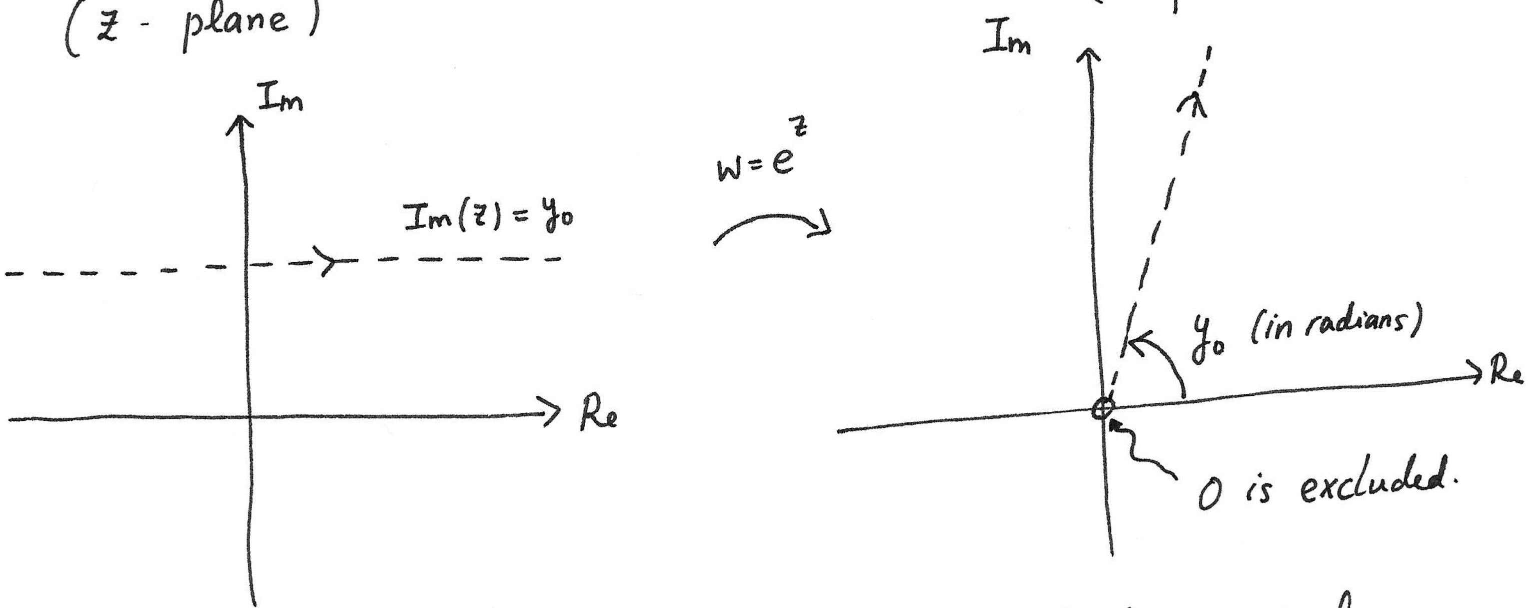
Periodicity: $\quad e^{z + 2\pi i} = e^z$, and

$$e^{z_1} = e^{z_2} \iff e^{z_1 - z_2} = 1 \iff z_1 - z_2 \in 2\pi i \mathbb{Z}.$$

(2.6) "Graph" of $w = e^z$. One way to visualize what $z \mapsto e^z$ transformation does, is to draw a few lines in the complex plane (where z lives) and sketch their images under e^z.

$$W = e^z = e^{Re(z)} \cdot e^{i \cdot Im(z)}$$





[Image of a line passing through 0 is going to be a spiral passing through 1 - try to sketch it if you want.]

(2.7) Sine and Cosine. From Euler's formula, we get

$$\left.\begin{aligned} e^{i\theta} &= \cos(\theta) + \sin(\theta)\, i \\ e^{-i\theta} &= \cos(\theta) - \sin(\theta)\, i \end{aligned}\right\} \Rightarrow \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \; ; \; \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} .$$

Define - for any $z \in \mathbb{C}$,

$$\boxed{\begin{aligned} \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \; ; \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}}$$

This allows us to prove all trigonometric identities algebraically.

Example.: $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$.

Proof Right-hand side $= \dfrac{(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) + (e^{iz} + e^{-iz})(e^{iw} - e^{-iw})}{4i}$

$$= \frac{e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{-i(z-w)}} - e^{-i(z+w)} + e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{-i(z-w)}} - e^{-i(z+w)}}{4i}$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w) \qquad \square$$

Check: (i) $\cos(-z) = \cos(z)$; $\sin(-z) = -\sin(z)$.

(ii) $\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$; $\cos(z + \pi) = -\cos(z)$

(iii) $\cos^2(z) + \sin^2(z) = 1$.