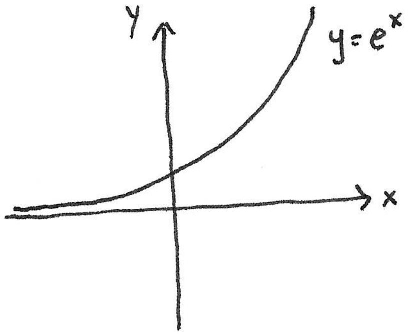
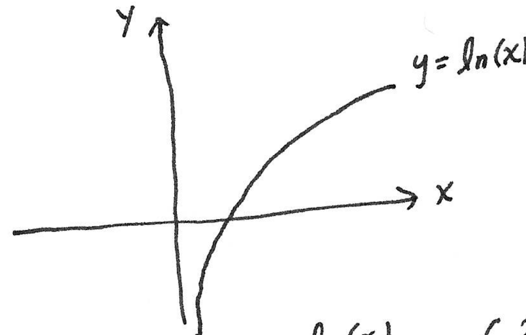
Lecture 3

(3.0) In this lecture, we are going to define logarithm and power function over complex numbers. Recall that for real numbers, natural logarithm is defined as the inverse to exponential function

$$b = \ln(a) \iff e^b = a \qquad (a \in \mathbb{R}_{>0}, \ b \in \mathbb{R})$$

That is, $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ is inverse to $e^{(\cdot)} : \mathbb{R} \to \mathbb{R}_{>0}$.





This allows us to define the powers $x^a = e^{a\ln(x)}$ $\left(\begin{matrix} x \in \mathbb{R}_{>0} \\ a \in \mathbb{R}\end{matrix}\right)$.

Properties of $\ln$: $\ln(1) = 0$ (because $e^0 = 1$)

$\ln(ab) = \ln(a) + \ln(b)$ (because $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$)

$$e^{\ln(x)} = x = \ln(e^x)$$

(3.1) <u>The</u> issue in defining logarithm of a complex number is that $z \mapsto e^z$ is no longer a one-to-one function. (see §2.5).

e.g. $$1 = e^0 = e^{2\pi i} = e^{4\pi i} = \cdots$$

$$\text{"log"}(1) = 0 \text{ or } 2\pi i \text{ or } 4\pi i \ \cdots$$ which value should we take?

As mentioned previously (see Lecture 0, page 4) - the ambiguity in defining "log"(z) is the same one we encounter in defining argument of z. To see this, note: for $z \in \mathbb{C}$; $z \neq 0$ (e^{α} is never zero; so "log"(0) will not be defined)

$$z = r \cdot e^{i\theta} = e^{\ln(r) + i\theta} \qquad \left(r \in \mathbb{R}_{>0}\ ;\ \theta = \arg(z) + 2\pi k \atop (k \in \mathbb{Z})\right)$$

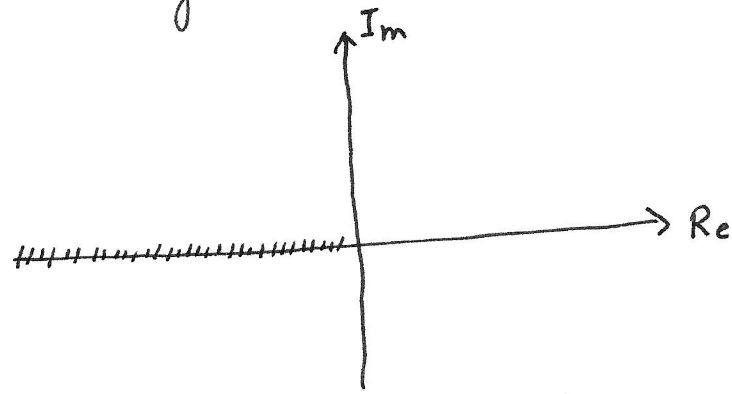
$$\Rightarrow \text{Logarithm of } z = \ln(r) + i\theta$$
$$= \ln(|z|) + i\,\mathrm{Arg}(z).$$

(3.2) <u>Principal value</u>, or <u>standard determination</u> of logarithm is defined on a "cut plane"

$$\log : \underbrace{\mathbb{C} - \mathbb{R}_{\leq 0}}_{\{z \in \mathbb{C} \text{ s.t. } z \text{ is not a negative real, or zero}\}} \longrightarrow \mathbb{C}$$

by the formula

$$\log(z) = \ln(|z|) + i\arg(z) \qquad (-\pi < \arg(z) < \pi)$$



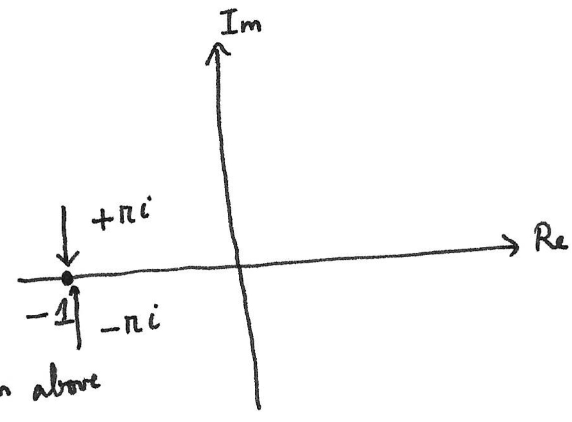
domain of definition of log

(3.3) Remarks. - (i) Some authors (including the ones of the problem book we are using) use Log (with capital L) for

$$\mathrm{Log}(z) = \ln(|z|) + i(\arg(z) + 2\pi k) \; ; \; k \in \mathbb{Z}.$$

as the "totality" of all possible logarithms. In literature the term "multivalued function" appears for such expressions. [This is obviously an absurd expression - a function, by definition, gives one output]. The theory of Riemann* surfaces was developed to allow one to view these "multi-valued functions" as honest functions (whose domain will not be a subset of $\mathbb{C}$, but a Riemann surface). This topic is beyond the scope of our course.

(ii) The reason the negative real axis is omitted from the definition of $\log : \mathbb{C} - \mathbb{R}_{\leq 0} \to \mathbb{C}$, is because $\arg(z) \in (-\pi, \pi]$ has discontinuity along $\mathbb{R}_{\leq 0}$.



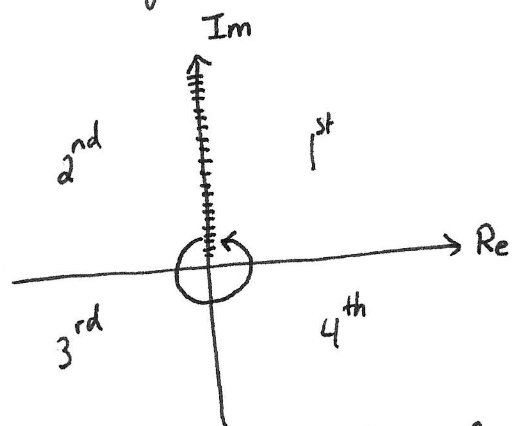
log of -1 would be

$$\begin{cases} +\pi i & \text{if approached from above} \\ -\pi i & \text{" " " below} \end{cases}$$

* Georg Friedrich Bernhard Riemann (17/9/1826 - 20/7/1866)

(iii) In general the "cut" along $\mathbb{R}_{\leq 0}$ could be made along any ray shooting off 0 (or even along any simple curve starting at 0 and going off to infinity). These different cuts give rise to different "branches of logarithm". All we can say for sure is that any two branches of logarithm will differ by $2\pi i \mathbb{Z}$.

For example, assume that we make a cut along positive imaginary direction - and define



$$\underline{\log} : \mathbb{C} \setminus \mathbb{R}_{\geq 0}\, i \longrightarrow \mathbb{C}$$

$$\underline{\log}(z) = \ln(|z|) + i\theta$$

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}.$$

(a different cut plane for a different logarithm)

Then, $\log(z) = \underline{\log}(z)$ for z lying in 1^{st}, 3^{rd} & 4^{th} quadrants

$\log(z) = \underline{\log}(z) + 2\pi i$ for z lying in 2^{nd} quadrant.

(3.4) Examples

$$\left(\begin{aligned} \operatorname{Log}(z) &= \ln(|z|) + i\left(\arg(z) + 2\pi k\right) \quad k \in \mathbb{Z} \\ \log(z) &= \ln(|z|) + i \arg(z)\ ; \quad -\pi < \arg(z) < \pi \end{aligned} \right)$$

[Later in the course we will prove that there is no logarithm defined on the entire complex plane with 0 omitted.]

- $\mathrm{Log}(-1) = (2k+1)\pi i \qquad (k \in \mathbb{Z})$
- $\mathrm{Log}(i) = \left(\frac{\pi}{2} + 2\pi k\right) i = (4k+1)\frac{\pi}{2} i \qquad (k \in \mathbb{Z})$
- $\log(i) = \frac{\pi}{2} i \qquad \left(\text{recall } i = e^{\frac{\pi}{2} i}\right).$
- $1 + i = \sqrt{2} \cdot e^{\frac{\pi}{4} i} \Rightarrow \log(1+i) = \ln(\sqrt{2}) + \frac{\pi}{4} i$

(3.5) Properties of logarithm.

- $e^{\mathrm{Log}(z)} = z$ (for any choice of logarithm) $(z \in \mathbb{C};\ z \neq 0)$.

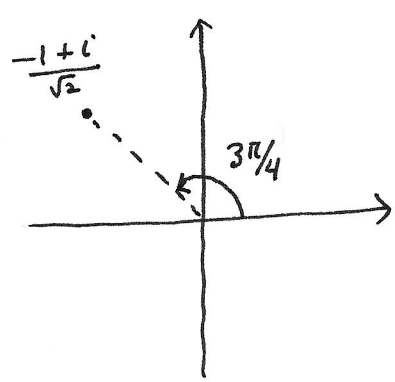
$$\left[\ \mathrm{Log}(z) = \ln(|z|) + (\arg(z) + 2\pi k) i \Rightarrow e^{\mathrm{Log}(z)} = e^{\ln(|z|)} \cdot e^{i\arg(z)} \cdot e^{2\pi k i} = |z| \cdot e^{i\arg(z)} = z. \right]$$

- $\mathrm{Log}(z_1 z_2)$ and $\mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ differ by $2\pi i n$ (for some $n \in \mathbb{Z}$).

(Since we restricted the domain of the standard / principal $\log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$; $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ is NOT true, and could even be meaningless, e.g. if $z_1 = z_2 = i$; $z_1 z_2 = -1$ is not in our domain.)

Example. $z = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = e^{\frac{3\pi}{4} i}$

$z^2 = e^{\frac{3\pi}{2} i} = -i$



$$\log(z^2) = -\frac{\pi}{2} i \neq 2\log(z) = \frac{3\pi}{2} i$$

(3.6) Let $\alpha \in \mathbb{C}$. Now we can define

$$z^{\alpha} = e^{\alpha \log(z)}$$ (same subtlety as for log should be kept in mind!)

Example Find all possible values of 2^i.

$$2^i = e^{i \log(2)} . \quad \text{Log}(2) = \ln(2) + 2\pi i k \quad (k \in \mathbb{Z})$$

(all possible values of $\text{Log}(2)$)

$$\Rightarrow 2^i = e^{i(\ln(2) + 2\pi i k)} = e^{-2\pi k} \cdot e^{i \ln(2)} \quad (k \in \mathbb{Z})$$

Example. Find all possible values of $(1+\sqrt{3}\, i)^{1/8}$.

$$1 + \sqrt{3}\, i = 2 \cdot e^{i(\frac{\pi}{3} + 2\pi k)} \Rightarrow \text{Log}(1 + \sqrt{3}\, i) = \ln(2) + i\left(\frac{\pi}{3} + 2\pi k\right)$$

$(k \in \mathbb{Z})$

$$\Rightarrow (1+\sqrt{3}\, i)^{\frac{1}{8}} = e^{\frac{1}{8}(\ln(2) + i(\frac{\pi}{3} + 2\pi k))}$$

(to simplify, we will write it as $\exp\left(\frac{1}{8}(\ln(2) + i(\frac{\pi}{3} + 2\pi k))\right)$.)

$$= e^{\frac{1}{8}\ln(2)} \cdot e^{i\left(\frac{\pi}{24} + \frac{2\pi}{8}k\right)}$$

$$= 2^{1/8} \cdot e^{\frac{\pi}{24} i} \cdot e^{\frac{2\pi k}{8} i} \quad (k = 0, 1, 2, \ldots, 7)$$

<u>Note</u> if $n \in \mathbb{Z}$, z^n has only one possible value with this definition – namely $\underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ times}}$ if $n > 0$

$$\frac{1}{\underbrace{z \cdots z}_{-n \text{ times}}} \quad \text{if } n < 0$$

Standard determination of $z^{\alpha} = \exp(\alpha \cdot \log(z))$ is defined using $\log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$. Therefore, its domain of definition is again restricted to $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The exception is when $\alpha \in \mathbb{Z}$.

	$\alpha \in \mathbb{Z}, \alpha \geq 0$;	$\alpha \in \mathbb{Z}, \alpha < 0$;	$\alpha \notin \mathbb{Z}$
Domain of z^{α} : (standard determination)	$\mathbb{C}$	$\mathbb{C} \setminus \{0\}$	$\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Example. Compute the "standard" value of i^i.

$$i = e^{\frac{\pi}{2} i} \Rightarrow \log(i) = \frac{\pi}{2} i.$$

So, $i^i = \exp\left(i \cdot \frac{\pi}{2} \cdot i\right) = e^{-\frac{\pi}{2}}$.

(3.7) The main topic of this course is <u>functions of a complex variable</u>. We have encountered the following functions already

z^n $(n \in \mathbb{Z})$, $\alpha \cdot z$ $(\alpha \in \mathbb{C}$ fixed$)$ $\rightsquigarrow$ Polynomials in z

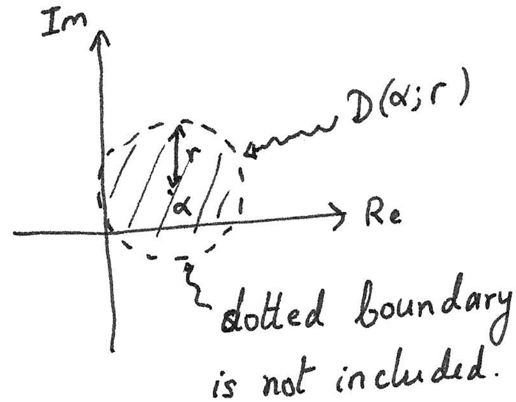
e^z, $\sin(z)$, $\cos(z)$, $\log(z)$, z^{α} $(\alpha \in \mathbb{C})$.

In general we will consider functions defined on a subset $\Omega \subseteq \mathbb{C}$. The subset will be assumed to be <u>open</u> and <u>connected</u>

(definitions - next lecture)

For now we will have to get used to subsets of $\mathbb{C}$ (or curves in $\mathbb{C}$) defined by inequalities (or equalities) featuring $z, \bar{z}$.

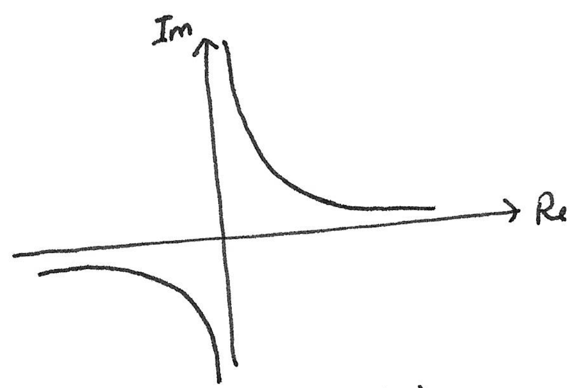
e.g. • $D(\alpha; r) = \{ z \in \mathbb{C} \mid |z - \alpha| < r \}$ = open disc around α, of radius r



• $Re(z) = -1$ defines the vertical line at $x = -1$.

• $Im(z^2) = 1$ translates to (for $z = x + yi$)

$2xy = 1$ which is a <u>hyperbola</u>



$\left(x \cdot y = \frac{1}{2} \right)$

• Let $\alpha, \beta \in \mathbb{C}$; $d \in \mathbb{R}$ such that $d > |\alpha - \beta|$. The set $\{ z \in \mathbb{C} : |z - \alpha| + |z - \beta| \leq d \}$ is the interior of the ellipse with foci at α and β. (boundary included)