Lecture 4

(4.0) Recall that our main objective in this course is to study functions of a complex variable : $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. We will always take Ω to be open and connected. Today's lecture contains definitions of these topological concepts.

(4.1) <u>Notation for discs</u>. Given $\alpha \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, define

$$D(\alpha; r) = \{ z \in \mathbb{C} : |z - \alpha| < r \}$$ - (open) disc of radius r, centered at α.

$$D^{\times}(\alpha; r) = \{ z \in \mathbb{C} : 0 < |z-\alpha| < r \} = D(\alpha; r) \smallsetminus \{\alpha\}$$

(punctured disc of radius r, centered at α)

$$\overline{D}(\alpha; r) = \{ z \in \mathbb{C} : |z-\alpha| \leq r \}$$ - (closed) disc.

(do not confuse $\overline{D}$ with complex conjugation)

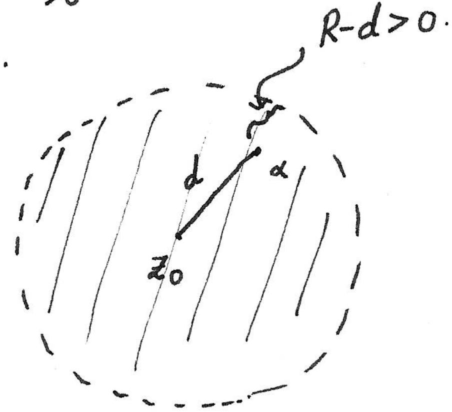
(4.2) Definition (1) : A subset $\mathcal{U} \subseteq \mathbb{C}$ is said to be <u>open</u> if for every $\alpha \in \mathcal{U}$, there exists $r \in \mathbb{R}_{>0}$ such that $D(\alpha; r) \subset \mathcal{U}$.

- In logical symbols : $\forall \alpha \in \mathcal{U}, \exists r \in \mathbb{R}_{>0}$ s.t. $D(\alpha; r) \subset \mathcal{U}$.
- $\mathcal{U} = \phi$ empty set is considered to be open.

This definition signifies that any point in an open set $\mathcal{U}$ can be approached along an arbitrary direction, while still staying in $\mathcal{U}$. This is a crucial requirement for studying differentiability of a function defined on $\mathcal{U}$.

Definition (2) : A subset $A \subseteq \mathbb{C}$ is said to be <u>closed</u> if its complement is open (i.e. $A^c := \mathbb{C} - A$ is open).

(4.3) Examples. (1) Let $z_0 \in \mathbb{C}$ and $R \in \mathbb{R}_{>0}$. Then $D(z_0; R)$ is open. To prove this, let $\alpha \in D(z_0; R)$.



Then $d = |z_0 - \alpha| < R$.

Choose r ; $0 < r < R-d$. We claim that $D(\alpha; r) \subset D(z_0; R)$.

Proof of the claim : if $\beta \in D(\alpha; r)$, then

$$|z_0 - \beta| = |z_0 - \alpha + \alpha - \beta| \leq |z_0 - \alpha| + |\alpha - \beta|$$
$$< d + r < R.$$

Hence, $\beta \in D(z_0; R)$. $\square$

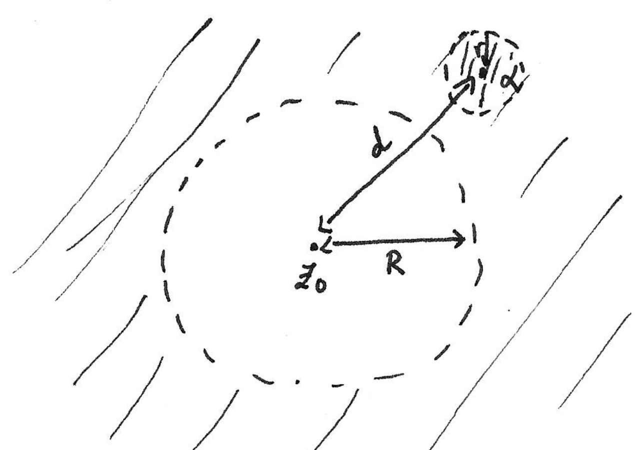
(2) $\overline{D}(z_0; R)$ is closed. Since $\mathbb{C} \setminus \overline{D}(z_0; R) = \{|z - z_0| > R\}$, we need to show that $|z - z_0| > R$ defines an open set.

Proof. Let $\alpha \in \mathbb{C} \setminus \overline{D}(z_0; R)$, i.e.

$$d = |z_0 - \alpha| > R.$$

Take $0 < r < d-R$. We have to show that $D(\alpha; r) \subset \mathbb{C} \setminus \overline{D}(z_0; R)$, that is

(To show) $|\alpha - \beta| < r \Rightarrow |z_0 - \beta| > R$.

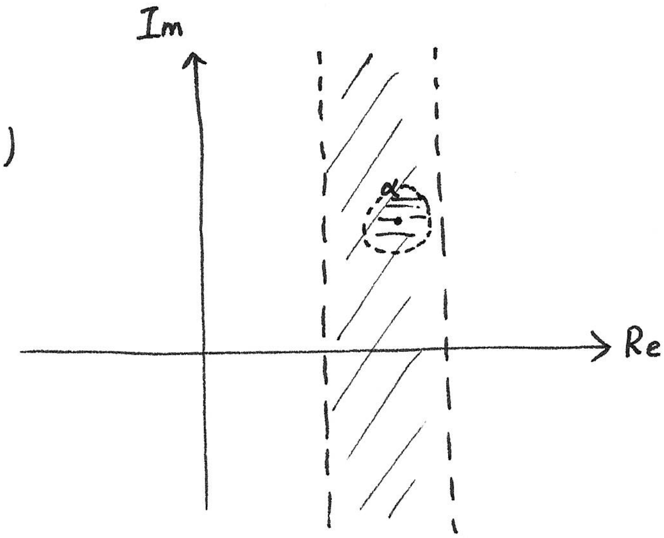


Complement of $\overline{D}(z_0; R)$

We can check this by contradiction. If $|z_0-\beta| \leq R$, then

$$d = |z_0-\alpha| \leq |z_0-\beta| + |\beta-\alpha| < R+r < d. \quad \text{Contradiction.} \quad \square$$

(3) $\mathcal{U} = \{1 < \mathrm{Re}(z) < 2\}$ is open.

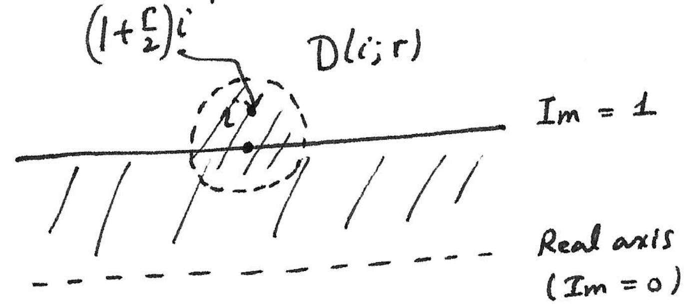


Let $\alpha \in \mathcal{U}$. If $\cancel{\alpha = \mathbb{R}}$ $a = \mathrm{Re}(\alpha)$ then $1 < a < 2$. Take $r > 0$ to be smaller than $a-1$ and $2-a$.

$$0 < r < \mathrm{Min}(a-1, 2-a).$$

Check: $D(\alpha; r) \subset \mathcal{U}$.

(4) $A = \{0 < \mathrm{Im}(z) \leq 1\}$ is neither open, nor closed.



- A is not open.

Pick $\alpha \in \mathbb{C}$ on $\mathrm{Im} = 1$ line, for example $\alpha = i$.

Claim: for every $r \in \mathbb{R}_{>0}$, $D(\alpha; r) \not\subset A$.

Proof. Let $r \in \mathbb{R}_{>0}$. Then $(1+\frac{r}{2})i \in D(i; r)$ but not in A. $\square$

- A is not closed. That is, we need to show

$$A^c = \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0 \text{ or } \mathrm{Im}(z) > 1\} \text{ is not open}$$

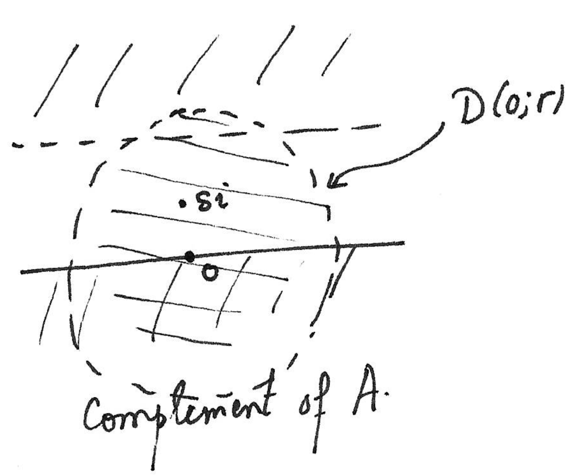
So, pick α on the real line, say $\alpha = 0$.

Let $r > 0$. As before we must produce an element of $D(0; r)$ not in A^c.

Take $0 < s < \mathrm{Min}(r, 1)$.

Then $si \in D(0; r)$, $si \notin A^c$.

$\square$



(5) ϕ and $\mathbb{C}$ are both open and closed.

(4.4) Proposition. (i) Let $\{\mathcal{U}_j\}_{j\in J}$ be a set of open subsets of $\mathbb{C}$ (J is just an indexing set). Then $\mathcal{U} = \bigcup_{j\in J} \mathcal{U}_j$ is again open.

[In words: (arbitrary) union of open sets is open.]

(ii) Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ be a finite collection of open subsets of $\mathbb{C}$. ($n \in \mathbb{Z}$; $n \geq 1$). Then $\mathcal{U} = \bigcap_{j=1}^{n} \mathcal{U}_j$ is open.

[In words: finite intersection of open sets is open.]

(Taking complements we get - (i) Arbitrary intersection of closed sets is closed; (ii) finite union of closed sets is closed.)

Proof. (i) If $\mathcal{U} = \phi$ (i.e. each $\mathcal{U}_j = \phi$) there is nothing to prove. Otherwise, let $\alpha \in \mathcal{U}$. That means there exists a $j_0 \in J$ such that $\alpha \in \mathcal{U}_{j_0}$. As $\mathcal{U}_{j_0}$ is open, there exists $r>0$ such that $D(\alpha;r) \subset \mathcal{U}_{j_0}$. Hence, $D(\alpha;r) \subset \bigcup_{j\in J} \mathcal{U}_j = \mathcal{U}$, proving that $\mathcal{U}$ is open.

(ii) Again it is enough to consider the case when $\bigcap_{j=1}^{n} \mathcal{U}_j \neq \phi$.

Let $\alpha \in \mathcal{U} = \bigcap_{j=1}^{n} \mathcal{U}_j$. That is, $\alpha \in \mathcal{U}_j$ for each $j = 1, \ldots, n$.

$\Rightarrow$ there exists $r_j \in \mathbb{R}_{>0}$ s.t. $D(\alpha; r_j) \subset \mathcal{U}_j$ $(j = 1, \ldots, n)$.

Take $r = \text{Min}\{r_1, \ldots, r_n\}$. Then $D(\alpha;r) \subset D(\alpha;r_j) \subset \mathcal{U}_j$ $\forall\, j = 1, \ldots, n$

$\Rightarrow D(\alpha;r) \subset \bigcap_{j=1}^{n} \mathcal{U}_j = \mathcal{U}$. Hence, $\mathcal{U}$ is open. $\square$

(4.5) Remark*. - The abstract definition of a "topological space" is modeled on the property established above. Namely, a topological space is a set X, together with a set of subsets of X, $\mathcal{T} \subset 2^X$ (= set of all subsets of X), such that (0) $\phi \in \mathcal{T}$; $X \in \mathcal{T}$.

(1) If $\{\mathcal{U}_j \in \mathcal{T}\}_{j \in J}$ then $\bigcup_{j \in J} \mathcal{U}_j \in \mathcal{T}$.

(2) If $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{T}$, then $\bigcap_{j=1}^{n} \mathcal{U}_j \in \mathcal{T}$.

(4.6) For $z_0 \in \mathbb{C}$, the singleton $A = \{z_0\} \subset \mathbb{C}$ is a closed set. Using the finite union property of closed sets, any finite subset $A \subset \mathbb{C}$, $|A| < \infty$ is closed.

Arbitrary intersections of open sets need not be open. For example, let $\mathcal{U}_n = D(0; \frac{1}{n})$ = (open) disc of radius $\frac{1}{n}$ centered at 0 $(n \geq 1)$. Then $\bigcap_{n=1}^{\infty} \mathcal{U}_n = \{0\}$ is not open.

(4.7) Definition. - A subset $A \subseteq \mathbb{C}$ is said to be path connected if for every $\alpha, \beta \in A$, there exists a continuous function

$$\gamma : [0,1] \longrightarrow A$$

such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$.

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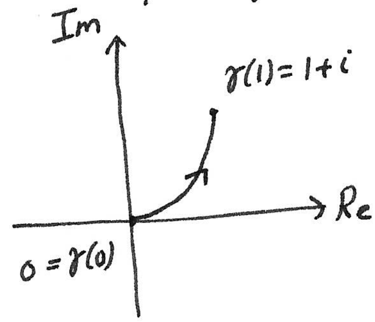
* Optional

- A <u>continuous function</u> $\gamma : [0,1] \to \mathbb{C}$ is called a <u>path</u> (or a <u>parametric curve</u>) joining $\gamma(0)$ and $\gamma(1)$. Writing real and imaginary parts, $\gamma(t) = x(t) + y(t)i$, and continuity refers to both <u>$x(t)$ and $y(t)$ being continuous</u>.

e.g. • $\gamma(t) = t + t^2 i \quad (0 \le t \le 1)$ is part of the parabola $y = x^2$ joining 0 to $1+i$



• $\gamma(t) = e^{it} \quad (0 \le t \le 2\pi)$ is the unit circle $|z| = 1$ traversed counterclockwise going 1 to itself

(4.8)
Formal definition of continuity of $\gamma : [0,1] \to \mathbb{C}$. – γ is said to be <u>continuous at $t_0 \in [0,1]$</u> if for every $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\begin{matrix} t \in [0,1] \\ |t - t_0| < \delta \end{matrix} \quad \text{implies} \quad |\gamma(t) - \gamma(t_0)| < \varepsilon .$$

γ is continuous if it is continuous at every $t_0 \in [0,1]$.

Remark. – Using the inequalities (for $\gamma(t) = x(t) + y(t)i$):

$$|\gamma(t) - \gamma(t_0)| = |(x(t) - x(t_0)) + (y(t) - y(t_0))i| \le |x(t) - x(t_0)| + |y(t) - y(t_0)|$$

and $\quad |x(t) - x(t_0)|, \ |y(t) - y(t_0)| \le \sqrt{(x(t) - x(t_0))^2 + (y(t) - y(t_0))^2}$

one can easily show that γ is continuous $\Leftrightarrow$ $x(t)$ and $y(t)$ are.

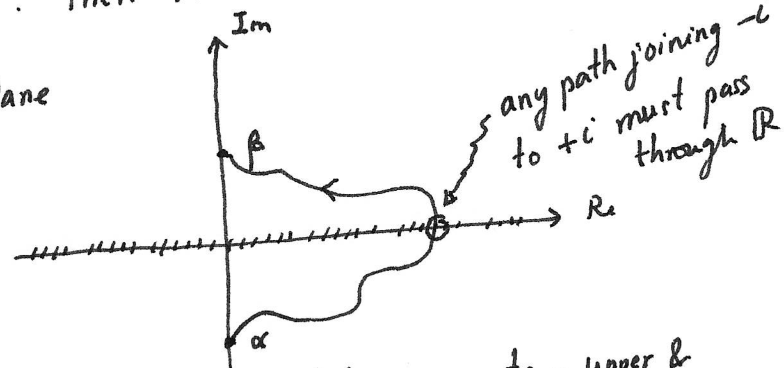
Remark.- The definition of continuity of $\gamma : [0,1] \to \mathbb{C}$ is equivalent to
$$\lim_{\substack{t \to t_0 \\ (0 \le t \le 1)}} \gamma(t) = \gamma(t_0)$$
[review the definition of limit].

(4.9) Example. Let $A = \mathbb{C} \setminus \mathbb{R}$. Then A is disconnected.

Proof.- Pick α in the lower half plane (i.e. $\mathrm{Im}(\alpha) < 0$)

and β in the upper half plane (i.e. $\mathrm{Im}(\beta) > 0$)

e.g $\alpha = -i$, $\beta = i$.



(A has 2 connected components - upper & lower half planes)

__Claim.__ There is no continuous $\gamma : [0,1] \to A$ s.t. $\gamma(0) = -i$, $\gamma(1) = +i$.

Proof of the claim: Assume $\gamma : [0,1] \to \mathbb{C}$ is any path joining $-i$ to i. Then $y(t) = \mathrm{Im}(\gamma(t))$ is a continuous function $[0,1] \to \mathbb{R}$ with $y(0) = -1$ and $y(1) = +1$. By intermediate value theorem, there must exist $t_0 \in (0,1)$ s.t. $y(t_0) = 0$. That is, $\gamma(t_0) \in \mathbb{R}$. $\square$

(4.10) More examples of connected (via paths) subsets.

$\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, $\mathbb{C} \setminus \mathbb{R}_{\le 0}$, $\mathbb{R}$, $D(z_0; R)$,

$D^{\times}(z_0; R)$, $\overline{D}(z_0; R)$, $\mathbb{C} \setminus D(z_0; R)$ [here $z_0 \in \mathbb{C}$ & $R \in \mathbb{R}_{>0}$].

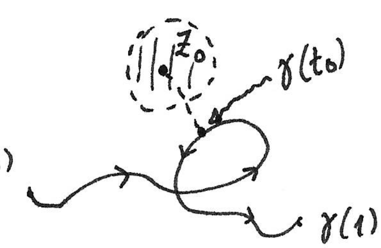
are all connected (via paths).

A subset $S \subset \mathbb{C}$ ($S \neq \phi$) is called __convex__ if for any $\alpha, \beta \in S$, the line segment joining α & β is in S. That is $t\alpha + (1-t)\beta \in S$ for every $0 \le t \le 1$. Every convex subset of $\mathbb{C}$ is clearly (path) connected.

(4.11) Remark*. – There is a more general notion of "connectedness" from topology (path-connected as defined in (4.7) above implies connected, but converse is not true, in general). For the sake of mentioning it - here is the general definition. A subset $A \subset \mathbb{C}$ is said to be disconnected if there exist two open sets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{C}$ such that (i) $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$, (ii) $A \cap \mathcal{U}_1 \neq \phi$, $A \cap \mathcal{U}_2 \neq \phi$ and (iii) $A \subset \mathcal{U}_1 \cap \mathcal{U}_2$. (e.g. in the example of (4.9), take $\mathcal{U}_1 = \{\mathrm{Im}(z) > 0\}$ and $\mathcal{U}_2 = \{\mathrm{Im}(z) < 0\}$). A is connected if it is not disconnected.

For our purposes, the restrictive notion of path-connectedness will suffice. In fact, it is known that for open sets in $\mathbb{C}$, the two notions are equivalent.

(4.12) Example. Let $\gamma : [0,1] \to \mathbb{C}$ be a path in $\mathbb{C}$. Then its image $C = \{\gamma(t) : 0 \leq t \leq 1\} \subset \mathbb{C}$ is closed.



Proof. Let $z_0 \in \mathbb{C}$ be a point not on C.

Define $D(t) = |\gamma(t) - z_0|^2 \in \mathbb{R}_{\geq 0}$.

Then $D(t)$ is a continuous function $[0,1] \xrightarrow{D} \mathbb{R}_{\geq 0}$ (if $z_0 = x_0 + y_0 i$, $\gamma(t) = x(t) + y(t) i$; then $D(t) = (x(t) - x_0)^2 + (y(t) - y_0)^2$ is continuous – since $x(t)$ & $y(t)$ are).

By extreme value theorem, D attains its absolute min. for some $t_0 \in [0,1]$ say, $D(t_0) = d^2 > 0$ (since $\gamma(t_0) \neq z_0$). Choose $0 < r < d$ and check that

$$D(z_0; r) \subset \mathbb{C} \setminus C. \qquad \square$$

* Optional