Lecture 5

(5.0) Recall that in the previous lecture we defined open/closed subsets of $\mathbb{C}$.

- $\Omega \subset \mathbb{C}$ is open if for every $\alpha \in \Omega$, we can find $r>0$ s.t. $D(\alpha;r) \subset \Omega$ ($D(\alpha;r) = \{z \in \mathbb{C} : |z-\alpha| < r\}$). [$\phi$ = empty set is open].
- $A \subset \mathbb{C}$ is closed if $A^c = \mathbb{C} \setminus A$ is open.

We also introduced paths (or parametric curves) in $\mathbb{C}$ and defined (path) connected subsets of $\mathbb{C}$.

In this lecture we will continue with some more topological notions related to $\mathbb{C}$.

(5.1) Accumulation points and interior points. Let $A \subset \mathbb{C}$, and $\alpha \in \mathbb{C}$.

We say <u>α is an accumulation point of A</u> if for every $r>0$, $D^\times(\alpha;r) \cap A \neq \phi$. (recall: $D^\times(\alpha;r) = \{0<|z-\alpha|<r\}$ is the punctured disc).

In more words, for every $r>0$, $\exists\, z \in A$ ($z \neq \alpha$) s.t. $|z-\alpha|<r$.

We say <u>α is an interior point of A</u> if there exists $r>0$ such that $D(\alpha;r) \subset A$. In particular, $\alpha \in A$. Note: by definition, A is open $\Leftrightarrow$ every $\alpha \in A$ is an interior point.

Let $\mathrm{Acc}(A)$:= the set of accumulation points of A.

$\mathring{A}$:= the set of interior points of A.

(5.2) Examples. (i) $A = D(z_0;R)$ ($z_0 \in \mathbb{C}$, $R \in \mathbb{R}_{>0}$ fixed).

then $\mathrm{Acc}(A) = \overline{D}(z_0;R) = \{|z-z_0| \leq R\}$.

$\mathring{A} = A$.

(ii) $A = \{z_0\}$. Then $\mathrm{Acc}(A) = \mathring{A} = \phi$ empty set.

Lemma. Let $A \subset \mathbb{C}$. Then, A is closed if and only if $\mathrm{Acc}(A) \subset A$.

Proof. A is closed $\Longleftrightarrow$ $\mathbb{C} \setminus A$ is open

$\Longleftrightarrow$ for every $\alpha \notin A$, there exists $r > 0$ s.t. $D(\alpha; r) \cap A = \phi$.

$\Longleftrightarrow$ for every $\alpha \in \mathbb{C}$, $[\alpha \notin A \Rightarrow \alpha \notin \mathrm{Acc}(A)]$.

Hence A is closed $\Longleftrightarrow$ $\mathrm{Acc}(A) \subset A$. $\square$

(5.3) **Closure and boundary.** – Let $A \subset \mathbb{C}$. We define <u>the closure of A</u>, denoted by $\overline{A}$, to be $\overline{A} = A \cup \mathrm{Acc}(A)$.

Ex. Verify that $\overline{A}$ is closed. Moreover if $A \subset B$, and B is a closed set, then $\overline{A} \subset B$.

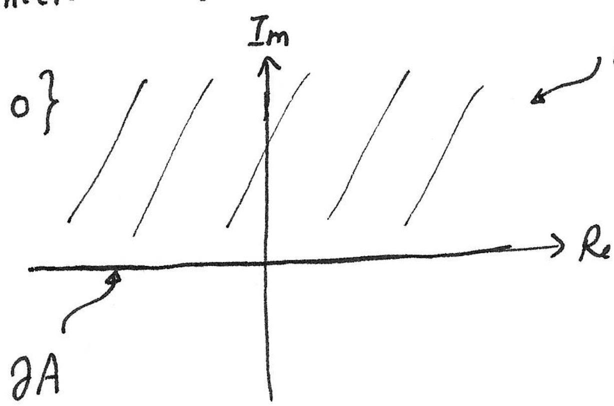
Warning: There is a bit of a clash of notations here. Do not confuse $\overline{A}$ with complex conjugation.

The <u>boundary of a set</u> $A \subset \mathbb{C}$ is defined as $\partial A := \overline{A} \setminus \mathring{A}$.

e.g. (i) $\partial D(z_0; R) = \partial \overline{D}(z_0; R) = \overline{D}(z_0; R) \setminus D(z_0; R) = \{z \in \mathbb{C} : |z - z_0| = R\}$

= circle of radius R centered at z_0.

(ii) Let $A = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$



$\overline{A} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$

$A = \mathring{A}$ (A is open).

$\partial A = \mathbb{R}$ (real axis).

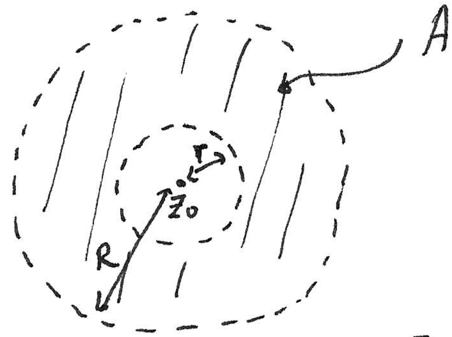
(iii) Let $0<r<R$ and $z_0 \in \mathbb{C}$. Define $A = \{ r < |z-z_0| < R \}$

A is open, so $A = \mathring{A}$.

$\overline{A} = \{ r \leq |z-z_0| \leq R \}$.

∂A consists of two components

$\{ |z-z_0| = r \} \cup \{ |z-z_0| = R \}$.



[∂A is disconnected]

(5.4) The definition of accumulation points can be given in terms of limits of sequences.

Definition. Let $\{z_1, z_2, \dots\} = \{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers. We say $\lim_{n\to\infty} z_n = l$ ($l \in \mathbb{C}$) if for every $\varepsilon > 0$, we can find $N > 0$ such that

$$|z_n - l| < \varepsilon \text{ for every } n > N.$$

Remark. (1) If $z_n = x_n + y_n i$ and $l = a + bi$, then

$$\lim_{n\to\infty} z_n = l \iff \lim_{n\to\infty} x_n = a \text{ and } \lim_{n\to\infty} y_n = b.$$

To see this, use the inequalities:

- $|x_n - a| \leq |z_n - l|$ and $|y_n - b| \leq |z_n - l|$
- $|z_n - l| = |(x_n - a) + i(y_n - b)| \leq |x_n - a| + |y_n - b|$ (by triangle ineq.)

(2) If $\lim_{n\to\infty} z_n = l$ and $\{z_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{z_n\}$ (i.e. $n_1 < n_2 < \cdots$)

Then $\lim_{k\to\infty} z_{n_k} = l$.

In view of this remark, we note that Cauchy's criterion* for convergence works over $\mathbb{C}$:

$$\lim_{n\to\infty} z_n \text{ exists} \iff \{z_n\}_{n=1}^{\infty} \text{ is a Cauchy sequence,}$$

i.e., for every $\varepsilon > 0$, we can find $N > 0$ s.t. $|z_n - z_m| < \varepsilon$, for every $n, m > N$.

(5.5) Properties of limits. (1) $\lim_{n\to\infty} z_n = \ell \Rightarrow \lim_{n\to\infty} \bar{z}_n = \bar{\ell}$, $\lim_{n\to\infty} |z_n| = |\ell|$.

(2) If $\lim_{n\to\infty} z_n = \alpha$ and $\lim_{n\to\infty} w_n = \beta$, then

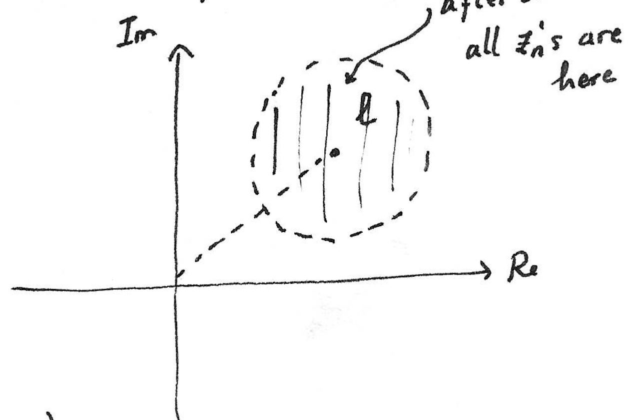
$$\lim_{n\to\infty} z_n w_n = \alpha\beta \quad \text{and} \quad \lim_{n\to\infty} (z_n + w_n) = \alpha + \beta.$$

(3) $\lim_{n\to\infty} z_n = \ell$, $c \in \mathbb{C}$ $\Rightarrow$ $\lim_{n\to\infty} c \cdot z_n = c \cdot \ell$.

(4) If $\lim_{n\to\infty} z_n = \ell \neq 0$, then there exists $N > 0$ such that $z_n \neq 0$ for every $n > N$ and $\displaystyle\lim_{\substack{n\to\infty \\ (n>N)}} \frac{1}{z_n} = \frac{1}{\ell}$.

Proof of (4) ((1), (2) and (3) are easy to check - left as an exercise).

Let $r = |\ell| > 0$. Take $\varepsilon = \frac{r}{2}$ to get $N > 0$ such that $|z_n - \ell| < \frac{r}{2}$ for every $n > N$.



Claim: $\forall n > N$; $|z_n| > \frac{r}{2}$.

(Pf. of the claim: $|z_n| = |\ell - (\ell - z_n)|$

$\geq \big||\ell| - |\ell - z_n|\big| > r - \frac{r}{2} = \frac{r}{2}$.)

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* Augustin-Luis Cauchy (21/8/1789 – 23/5/1857)

Now $(n > N)$: $\left| \frac{1}{z_n} - \frac{1}{\ell} \right| = \left| \frac{\ell - z_n}{z_n \ell} \right| = \frac{|z_n - \ell|}{|z_n| \cdot |\ell|}$

$$\leq \frac{2}{r^2} |z_n - \ell|.$$

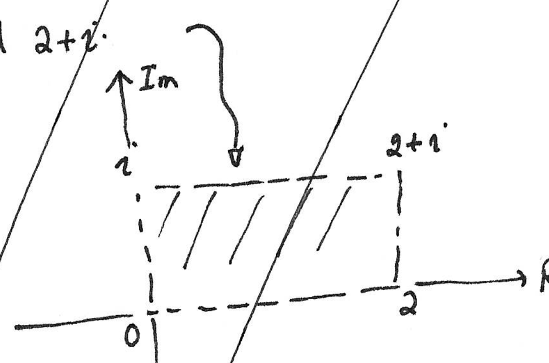
So, given $\varepsilon > 0$, choose $M > N$ s.t. for every $n > M$, $|z_n - \ell| < \frac{r^2}{2} \cdot \varepsilon$. □

<u>Remark</u>. (re-defining accumulation points): Let $A \subset \mathbb{C}$ and $\alpha \in \mathbb{C}$. Then α is an accumulation point of A if and only if we can find a sequence $\{z_n\}_{n=1}^{\infty} \subset A \setminus \{\alpha\}$ converging to α.

(Proof. Assume $\alpha \in Acc(A)$. For each $n \geq 1$, choose $z_n \in \overset{\times}{D}(\alpha; \frac{1}{n}) \cap A$. It is easy to check that $\lim_{n \to \infty} z_n = \alpha$. Conversely, if there is a sequence $\{z_n\}_{n=1}^{\infty} \subset A \setminus \{\alpha\}$ such that $\lim_{n\to\infty} z_n = \alpha$, then for every $r > 0$, we can find $N > 0$ s.t. $|z_n - \alpha| < r$ for all $n > N$. Hence $\overset{\times}{D}(\alpha; r)$ contains $\{z_{N+1}, z_{N+2}, \dots\} \subset A$ proving that $\alpha \in Acc(A)$. □)

(5.6) Bounded subsets of $\mathbb{C}$. Let $A \subset \mathbb{C}$. We say <u>A is bounded</u> if we can find $R > 0$ such that $|z| < R$ for every $z \in A$.

e.g. (i) $D(\alpha; r)$ is bounded. (ii) Let A be the part of the complex plane within the rectangle with vertices $0, 2, i$ and $2+i$. Then $|z| < \sqrt{5}$ for every $z \in A$. Hence A is bounded.



Next time

(iii) $\mathbb{Z} \subset \mathbb{C}$ is not bounded.