Lecture 6

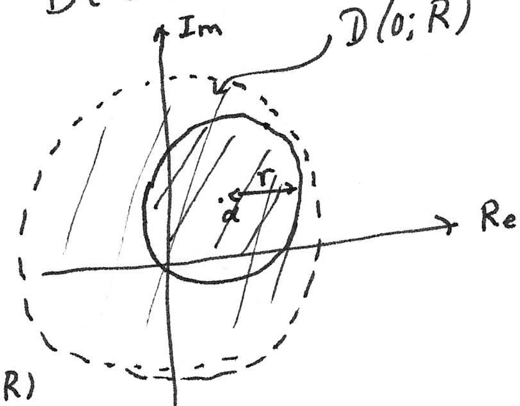
(6.0) We have been studying some topological notions for subsets of $\mathbb{C}$. So far we have defined: open/closed/(path)-connected sets; accumulation points and interior points of a set; closure and boundary of a set; sequences - convergence - Cauchy's criterion - limits.

Next concept we will need (for later) is that of a compact set.

(6.1) Bounded sets. - Let $A \subset \mathbb{C}$. We say <u>A is bounded</u> if ~~iff~~ there exists $R>0$ such that $|z| < R$ for every $z \in A$.

That is, $A \subset D(0;R)$ for some R.

e.g. (i) Let $\alpha \in \mathbb{C}$ and $r>0$. Then $\overline{D}(\alpha;r)$ is bounded.



($\forall z \in \overline{D}(\alpha;r)$; we have

$$|z| = |(z-\alpha)+\alpha| \leq |z-\alpha| + |\alpha|$$
$$\leq r + |\alpha|$$

Take $R > r+|\alpha|$. Then $D(\alpha;r) \subset D(0;R)$)

(ii) $A = \{z \in \mathbb{C} : 0 < \operatorname{Im}(z) < 1\}$ is not bounded (or, unbounded)

(for instance $x + \frac{1}{2}i \in A$ for every $x \in \mathbb{R}$ and $|x+\frac{1}{2}i| > |x|$.)

(iii) Let $\gamma : [0,1] \to \mathbb{C}$ be a path. $\mathcal{C} = \{\gamma(t) : 0 \leq t \leq 1\} \subset \mathbb{C}$ is a bounded set.

Proof. Consider $|\gamma|^2 : [0,1] \to \mathbb{R}$ - continuous function. By Extreme Value Theorem, $\exists\, M \in \mathbb{R}_{\geq 0}$ s.t. $|\gamma(t)|^2 < M$ for every $t \in [0,1]$ i.e. $\mathcal{C} \subset D(0;\sqrt{M})$. $\square$

(6·2) Theorem.- Let $A \subset \mathbb{C}$. Then A is closed and bounded if, and only if, for every sequence $\{z_n\}_{n=1}^{\infty}$ in A, we can find a convergent subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ with $\lim\limits_{k\to\infty} z_{n_k} \in A$. In other words, every infinite subset of A has accumulation points in A. (see Remark on page 5 of Lectures 5).

Remark.- This result is the complex analogue of the famous Bolzano-Weierstrass* Theorem: every bounded sequence of real numbers has a convergent subsequence. In fact, Bolzano-Weierstrass theorem is used in the proof of this theorem.

Proof. $(\Rightarrow)$ Assume that A is closed and bounded, and let $\{z_n = x_n + y_n i\}_{n=1}^{\infty}$ be a sequence in A. Then $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers (since $|x_n| \leq |z_n|$). By Bolzano-Weierstrass, we can find a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some $x \in \mathbb{R}$.

Now consider the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ - again bounded and hence has a convergent subsequence $\{y_{n_{k_\ell}}\}_{\ell=1}^{\infty}$. To avoid messy notation, set $\xi_\ell = x_{n_{k_\ell}}$, $\eta_\ell = y_{n_{k_\ell}}$

$\{\alpha_\ell = \xi_\ell + \eta_\ell i\}_{\ell=1}^{\infty}$ is now a convergent subsequence of $\{z_n\}_{n=1}^{\infty}$

$\left(\lim\limits_{\ell\to\infty} \xi_\ell = x, \quad \lim\limits_{\ell\to\infty} \eta_\ell = y\right)$ see Remark (2) on page 3 (Lectures 5)

$\lim\limits_{\ell\to\infty} \alpha_\ell = x + yi \in A$ because A is closed. (see Remark on page 5 - Lectures 5 and Lemma 5·2).

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* Bernard Bolzano (5/10/1781 - 18/12/1848); Karl Weierstrass (31/10/1815 - 19/2/1897)

($\Leftarrow$) We can easily check that A is closed - using Lemma (5.2).

Let $\alpha \in \mathbb{C}$ be an accumulation point of A. Then we can find a sequence in $A \setminus \{\alpha\}$ converging to α (Remark on Lecture 5 page 5). By our assumption $\alpha \in A$. Hence, $Acc(A) \subset A \Rightarrow A$ is closed.

To see that A is bounded, we argue by contradiction. Assume A is not bounded. This means, for every $n \geq 1$ ($n \in \mathbb{Z}$), we can find $z_n \in A$ with $|z_n| > n$. By our hypothesis, we can now take a convergent subsequence $\{z_{n_k}\}_{k=1}^{\infty}$, say $w = \lim_{k \to \infty} z_{n_k}$. Taking $\varepsilon = 1$ in the definition of the limit we conclude that there exists $M > 0$ s.t. $|z_{n_k} - w| < 1$ for every $n_k > M$. Meaning $|z_{n_k}| < |w| + 1$ contradicting the fact that $|z_{n_k}| > n_k$ and $n_1 < n_2 < \cdots$ become arbitrarily large $\square$

(6.3) Heine-Borel* Theorem. Recall, for $\mathbb{R}$, Heine-Borel theorem states that if $\{I_j\}_{j \in J}$ is a set of open intervals in $\mathbb{R}$ and $[a,b] \subset \bigcup_{j \in J} I_j$, then there exist $I_{j_1}, \ldots, I_{j_n}$ such that $[a,b] \subset \bigcup_{k=1}^{n} I_{j_k}$. Often written in the form "every open cover of $[a,b]$ has a finite subcover" - this property is taken as a definition of "compact spaces" in general topology.

We have the following analogue of Heine-Borel theorem over $\mathbb{C}$. <u>The proof given below is optional</u>, and <u>works for any metric space</u>.

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* Heinrich Eduard Heine (16/3/1821 - 21/10/1881); Émile Borel (7/1/1871 - 3/2/1956)

Theorem. Let $A \subset \mathbb{C}$. Then A is <u>closed and bounded</u> (or equivalently, by Thm ~~(5.6)~~ (6.2) - page ~~6~~ 2, every sequence in A has a subsequence converging to a point in A) if and only if <u>every open cover of A has a finite subcover</u>.

(Meaning - given a set of open sets in $\mathbb{C}$, $\{U_j\}_{j\in J}$, such that $A \subset \bigcup_{j\in J} U_j$, there exists finitely many $U_{j_1}, \dots, U_{j_n}$ such that $A \subset \bigcup_{k=1}^{n} U_{j_k}$).

Proof*- ($\Leftarrow$) Assume on the contrary that we have $\{z_n\}_{n=1}^{\infty} \subset A$ with no converging subsequence (limiting to a point in A). Then, for every $\alpha \in A$, α is not an accumulation point of $\{z_n\}_{n=1}^{\infty}$. That is, $\forall \alpha \in A$, $\exists\, r_\alpha > 0$ such that $\mathring{D}(\alpha; r_\alpha) \cap \{z_n\}_{n=1}^{\infty} = \phi$.

In other words, $\left| D(\alpha; r_\alpha) \cap \{z_n\}_{n=1}^{\infty} \right| = 0$ or 1 (if $\alpha = z_n$ for some n).

Take the open cover $\{D(\alpha; r_\alpha)\}_{\alpha\in A}$ of A. If $D(\alpha_1; r_{\alpha_1}), \dots, D(\alpha_N, r_{\alpha_N})$ is its finite subcover then $\{z_n\}_{n=1}^{\infty} \subset A \subset \bigcup_{k=1}^{N} D(\alpha_k, r_{\alpha_k})$ implies that $\{z_n\}_{n=1}^{\infty}$ is a finite set $\Rightarrow \exists\, z \in A$ s.t. $z_n = z$ for infinitely many n's

$\Rightarrow$ $\{z_n\}_{n=1}^{\infty}$ has a subsequence converging to $z \in \mathbb{C}$. Contradiction.

($\Rightarrow$) This part of the proof is rather long and is broken into two crucial steps.

<u>Assumption</u> : $A \subset \mathbb{C}$. for every $\{z_n\}_{n=1}^{\infty} \subset A$, there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} z_{n_k} \in A$.

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* Optional. This proof is due to Henri Lebasgue (28/6/1875 - 26/7/1941)

Step 1. Let $A \subset \bigcup_{j \in J} U_j$ be an open cover. Then we can find $r > 0$ such that: for every $\alpha \in A$, there exists $j \in J$ so that $D(\alpha; r) \subset U_j$.

[The supremum of all such r's is called the Lebasgue number of the open cover].

Step 2. Given any $\delta > 0$, we can find a finili set $\{\alpha_1, \dots, \alpha_N\} \subset A$ s.t.

$$A \subset \bigcup_{j=1}^{N} D(\alpha_j; \delta).$$

Assuming these two statements, the proof of $(\Rightarrow)$ proceeds as follows. Let $A \subset \bigcup_{j \in J} U_j$ be an open cover of A. Pick $r > 0$ as in Step 1. Take $\delta = r$ in Step 2 to find $\alpha_1, \dots, \alpha_N \in A$. According to Step 1, there exist $j_1, \dots, j_N \in J$ so that $D(\alpha_k; r) \subset U_{j_k}$ $(1 \le k \le N)$. Then $A \subset \bigcup_{k=1}^{N} D(\alpha_k; r) \subset \bigcup_{k=1}^{N} U_{j_k}$. □

Proof of Step 1. Assume the contrary. Taking $r = \frac{1}{n}$ $(n \ge 1)$, we have:

$\forall n \ge 1$, $\exists z_n \in A$ s.t. $D(z_n; \frac{1}{n})$ is not contained in any U_j $(j \in J)$.

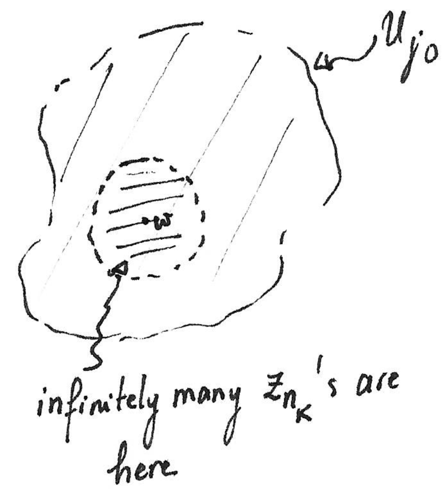
Using the hypothesis on A, there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ with $w = \lim_{k \to \infty} z_{n_k} \in A$. Let $j_0 \in J$ be such that $w \in U_{j_0}$. (since $A \subset \bigcup_{j \in J} U_j$).

As U_{j_0} is open, we can find $r_0 > 0$ so that $D(w; r_0) \subset U_{j_0}$

Choose $N > 0$ s.t. $\frac{1}{N} < \frac{r_0}{2}$; and $l > 0$ s.t. $n_l > N$ and $|z_{n_k} - w| < \frac{1}{N}$ for every $k > l$.

Then $D(z_{n_k}; \frac{1}{n_k}) \subset D(w; r_0) \subset U_{j_0}$

contradicting our choice of z_n's.



Proof of Step 2. Again we argue by contradiction. Assuming the contrary of the assertion of Step 2, there exists a $\delta > 0$ so that

$$A \not\subset D(\alpha_1;\delta) \cup \cdots \cup D(\alpha_N;\delta)$$ – for every finite number of elements $\alpha_1, \ldots, \alpha_N \in A$. Again we can construct a sequence in A, and prove that it has no convergent subsequence.

. Pick $z_1 \in A$, then $z_2 \in A \setminus D(z_1;\delta)$, $z_3 \in A \setminus (D(z_1;\delta) \cup D(z_2;\delta))$

and so on: $\quad z_{n+1} \in A \setminus \left(\bigcup_{j=1}^{n} D(z_j;\delta) \right)$.

If $\{z_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{z_n\}_{n=1}^{\infty}$, then there will exist $N > 0$ so that $|z_{n_k} - z_{n_\ell}| < \delta$ for every $n_k, n_\ell > N$.

This contradicts the choice of our sequence. □

~~(5.8)~~ (6·4) A set $K \subset \mathbb{C}$ satisfying either of the following three equivalent conditions is called <u>compact</u>. (Thms. (6.2) and (6.3)).

(1) K is closed and bounded.

(2) Every sequence in K has a subsequence converging to some point in K.

(3) Every open cover of K has a finite subcover.

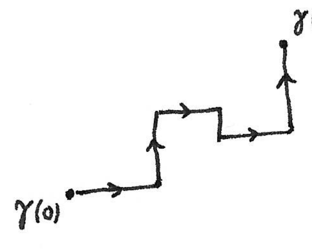
Remark.– (1) is easier to verify in concrete examples. Property (2) is used in obtaining analogues of the extreme value theorem. An important application of (3) is given below.

(6.5) <u>Zig-zag paths</u>.- A path $\gamma : [0,1] \to \mathbb{C}$ is zig-zag if it consists of finitely many <u>horizontal</u> or <u>vertical line segments</u>.

Meaning - there exist $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1 \quad (n \geq 1)$ such that for each j, $0 \leq j \leq n-1$, either $\mathrm{Im}(\gamma(x)) = \mathrm{Im}(\gamma(t_j)) \ \forall x \in [t_j, t_{j+1}]$

or $\mathrm{Re}(\gamma(x)) = \mathrm{Re}(\gamma(t_j)) \ \forall x \in [t_j, t_{j+1}]$

<u>Remarks on paths</u> :



(Example of a zig-zag path)

(i) We have chosen to take the domain of our paths to be $[0,1]$ so far. In general, it could be <u>any closed interval</u> $[a,b] \subset \mathbb{R} \quad (a < b)$.

(ii) Paths can be <u>concatenated</u> : if $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ are two paths such that $\gamma_1(b) = \gamma_2(b)$, we can define $\gamma : [a,c] \to \mathbb{C}$ as

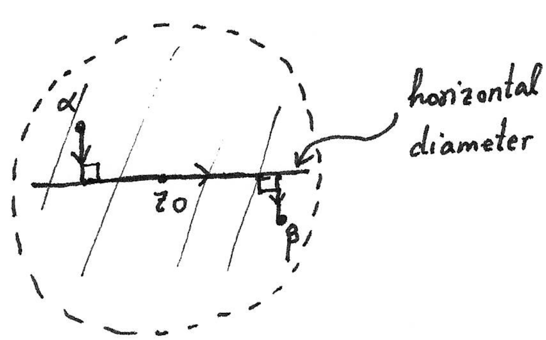
$$\gamma(t) = \begin{cases} \gamma_1(t), & a \leq t \leq b \\ \gamma_2(t), & b \leq t \leq c \end{cases}$$

- concatenation of γ_1 and γ_2, denoted by $\gamma_1 \cdot \gamma_2$.

<u>Proposition</u>.- Let $\Omega \subset \mathbb{C}$ be an open and (path)-connected set. Then any two points in Ω can be joined by a zig-zag path.

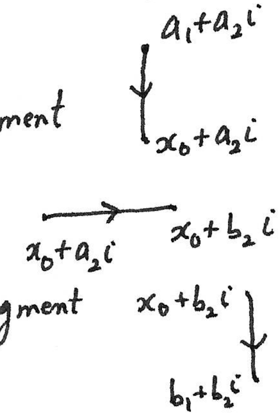
Proof - The idea is to prove it for an open disc first. Then, given any path $\gamma : [a,b] \to \Omega$, cover the image of γ by finitely many discs - still within Ω.

<u>Case when $\Omega = D(z_0; R)$</u>. Given $\alpha, \beta \in D(z_0; R)$ we can join them to the horizontal diameter via vertical line segments.



Explicitly, let $\alpha = a_1 + a_2 i$; $\beta = b_1 + b_2 i$; $z_0 = x_0 + y_0 i$

$\gamma : [0,3] \to D(z_0; R)$ is given by:



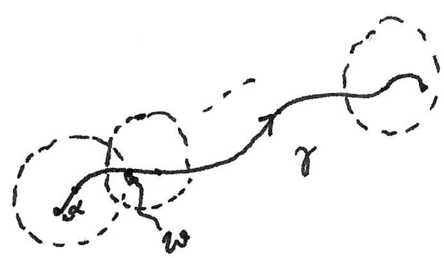
$$\gamma(t) = \begin{cases} a_1 + t(x_0 - a_1) + a_2 i \ ; \ 0 \leq t \leq 1 & \leftarrow \text{vertical segment} \\ x_0 + \left(a_2 + (t-1)(b_2 - a_2)\right) i \ ; \ 1 \leq t \leq 2 & \leftarrow \text{horizontal} \\ x_0 + (t-2)(b_1 - x_0) + b_2 i \ ; \ 2 \leq t \leq 3 & \leftarrow \text{vertical segment} \end{cases}$$

Now let $\Omega \subset \mathbb{C}$ be arbitrary open and connected subset. Let $\alpha, \beta \in \Omega$.
Let $\gamma : [0,1] \to \Omega$ be any path joining α to β. $\mathcal{C} = \{\gamma(t) : 0 \leq t \leq 1\}$ is closed and bounded. (see Lecture 4, Ex. (4.12) - page 8 ; and (6.1) - Example (iii) above).

For any $z \in \mathcal{C}$, we can find $r_z > 0$ s.t. $D(z; r_z) \subset \Omega$. (as Ω is open).

$\mathcal{C} \subset \bigcup_{z \in \mathcal{C}} D(z, r_z)$ $\Rightarrow$ we can find $z_1, \dots, z_N$ on $\mathcal{C}$ s.t. $\mathcal{C} \subset \bigcup_{j=1}^{N} D(z_j, r_{z_j})$

Renumbering, if necessary, assume that $\alpha \in D(z_1; r_1)$; $\beta \in D(z_N; r_N)$ and
$D(z_j; r_j) \cap D(z_{j+1}; r_{j+1}) \neq \phi$. , as follows:



- Choose z_1 s.t. $\alpha \in D(z_1; r_1)$.
- Let $\{w\} = \partial D(z_1; r_1) \cap \mathcal{C}$. More precisely,
  let $t_1 \in [0,1]$ s.t. $\gamma([0,t_1)) \subset D(z_1; r_1)$. $w = \gamma(t_1)$.
- Choose z_2 s.t. $w \in D(z_2; r_2)$ - continue until N.

$\square$

Now we are done by the Ω = open disc case