Lecture 7

(7.0) In this lecture we will start studying functions of a complex variable $f : A \to \mathbb{C}$ (where $A \subset \mathbb{C}$). Writing real and imaginary parts:

$$f(x+yi) = u(x,y) + i\,v(x,y)$$

makes it clear that f is completely determined by two real-valued functions of two real variables $u, v : A \to \mathbb{R}$.

For the notions of <u>limit and continuity</u> there is no difference between the complex version and real version (for function of 2 variables). as we will see below.

(7.1) <u>Limit of a function $f : A \to \mathbb{C}$.</u> Assume $A \subset \mathbb{C}$ ($A \neq \phi$) and $\alpha \in \mathrm{Acc}(A)$ (i.e., α is an accumulation point of A).

$$\boxed{\lim_{\substack{z \to \alpha \\ (z \in A)}} f(z) = w}$$ means $$\boxed{\begin{array}{l}\text{for every } \varepsilon > 0, \text{ we can find} \\ \delta > 0 \text{ so that} \\ \underset{(z \in A)}{0 < |z - \alpha| < \delta} \Rightarrow |f(z) - w| < \varepsilon\end{array}}$$

Recall that (Lecture 5 - page 5) α is an accumulation point of A implies (in fact, is equivalent to) we have a sequence $\{z_n\}_{n=1}^{\infty} \subset A - \{\alpha\}$ converging to α. Using this fact, we can rewrite:

<u>Lemma.</u>

$$\lim_{\substack{z \to \alpha \\ (z \in A)}} f(z) = w \iff \lim_{n \to \infty} f(z_n) = w, \text{ for every sequence } \{z_n\}_{n=1}^{\infty} \subset A \setminus \{\alpha\} \text{ s.t. } \lim_{n \to \infty} z_n = \alpha$$

Proof. ($\Rightarrow$) This is just a matter of unfolding the definitions.

Assume $\lim\limits_{\substack{z \to \alpha \\ (z \in A)}} f(z) = w$ and $\{z_n\}_{n=1}^{\infty} \subset A - \{\alpha\}$, $\lim\limits_{n \to \infty} z_n = \alpha$.

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that $\underset{(z \in A)}{|z - \alpha| < \delta} \Rightarrow |f(z) - w| < \varepsilon$ (from the definition of $\lim\limits_{z \to \alpha} f(z)$). Since $z_n \to \alpha$ as $n \to \infty$, we can find $N > 0$ so that $|z_n - \alpha| < \delta$ for every $n > N$. Hence $|f(z_n) - w| < \varepsilon$ for every $n > N$., i.e. $\lim\limits_{n \to \infty} f(z_n) = w$

($\Leftarrow$) Assume $\lim\limits_{n \to \infty} f(z_n) = w$ for every sequence $\{z_n\} \subset A - \{\alpha\}$ converging to α. Assume (for the sake of contradiction) that $\lim\limits_{\substack{z \to \alpha \\ (z \in A)}} f(z) \neq w$. This means that there is some $\varepsilon > 0$ for which no $\delta > 0$ could be found. : i.e.,

$\exists\, \varepsilon > 0$ such that for every $\delta > 0$ there is some $z \in A$ with $0 < |z - \alpha| < \delta$ and $|f(z) - w| \geq \varepsilon$.

For $n \geq 1$, take $\delta = \frac{1}{n}$ and let $z_n \in A$ $(z_n \neq \alpha)$ be s.t.

$0 < |z_n - \alpha| < \frac{1}{n}$ and $|f(z_n) - w| \geq \varepsilon$.

This sequence $\{z_n\}_{n=1}^{\infty}$ converges to α and $\lim\limits_{n \to \infty} f(z_n) \neq w$, contradicting our assumption. $\square$

(7.2) Basic properties. (see Lecture 5, §5.5 - page 4).

[$A \subset \mathbb{C}$; $\alpha \in Acc(A)$; $f, g : A \to \mathbb{C}$ two functions.]

(1) $\lim_{z \to \alpha} f(z) = \ell \Rightarrow \lim_{z \to \alpha} \overline{f(z)} = \overline{\ell}$, $\lim_{z \to \alpha} |f(z)| = |\ell|$.

(2) $\lim_{z \to \alpha} f(z) = \ell \iff \lim_{z \to \alpha} Re(f(z)) = Re(\ell)$ and

$\lim_{z \to \alpha} Im(f(z)) = Im(\ell)$

(3) $\lim_{z \to \alpha} f(z) = \ell_1$, $\lim_{z \to \alpha} g(z) = \ell_2$ and $a, b \in \mathbb{C}$ implies

$\lim_{z \to \alpha} f(z) \cdot g(z) = \ell_1 \ell_2$ and $\lim_{z \to \alpha} (a f(z) + b g(z)) = a\ell_1 + b\ell_2$.

(4) $\lim_{\substack{z \to \alpha \\ (z \in A)}} f(z) = \ell \neq 0$ implies there is a subset $B \subset A$; $\alpha \in Acc(B)$ such that $f(z) \neq 0 \ \forall z \in B$; and

$$\lim_{\substack{z \to \alpha \\ (z \in B)}} \frac{1}{f(z)} = \frac{1}{\ell}.$$

(7.3) Again, let $A \subset \mathbb{C}$; $f : A \to \mathbb{C}$ a function ; and $\alpha \in Acc(A)$.

We say f is <u>continuous at α</u> if $\boxed{\alpha \in A \text{ and } \lim_{\substack{z \to \alpha \\ (z \in A)}} f(z) = f(\alpha)}$

[<u>f is continuous</u> if <u>continuous at every α</u>.]

Remark.- We will mostly consider functions defined on an open set $\Omega \subset \mathbb{C}$, and investigate whether or not a given $f : \Omega \to \mathbb{C}$ extends to the boundary of Ω, i.e. does there exist continuous $\tilde{f} : \overline{\Omega} \to \mathbb{C}$ such that $\tilde{f}(z) = f(z)$ $\forall z \in \Omega$.

$\overline{\Omega}$: closure of Ω

The properties of limits of a function listed in §7.2 above immediately imply analogous properties of continuous functions.

$[A \subset \mathbb{C};\ f_1, f_2, f : A \to \mathbb{C}$ continuous functions$]$

(1) $\bar{f} : A \to \mathbb{C}$, and $|f| : A \to \mathbb{R}$ are again continuous.
$\bar{f}(z) = \overline{f(z)}$ $\qquad |f|(z) = |f(z)|$

(2) f is continuous $\iff$ $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are continuous

(3) $af_1 + bf_2 : A \to \mathbb{C}$ and $f_1 \cdot f_2 : A \to \mathbb{C}$ are continuous.
$(a, b \in \mathbb{C})$ $\quad z \mapsto af_1(z) + bf_2(z)$ $\qquad z \mapsto f_1(z) \cdot f_2(z)$

(4) $(f(z) \neq 0,\ \forall z \in A)$ $\quad \dfrac{1}{f} : A \to \mathbb{C}$ is again continuous.
$z \mapsto \dfrac{1}{f(z)}$

(5) Composition of continuous functions is continuous: let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two continuous functions. Assume $f(z) \in B, \forall z \in A$. Then $g \circ f : A \to \mathbb{C}$ defined as: $(g \circ f)(z) = g(f(z))$ (is called <u>Composition of f and g</u>) is also continuous. (Easy exercise: prove this.)

(7.4) Examples. – By property (2) above, a continuous function $f : A \to \mathbb{C}$ is given by two (real-valued) functions of two (real) variables.

(i) $f(x+yi) = xy + i \sin(x+y) : \mathbb{C} \to \mathbb{C}$ is continuous.

(ii) $f(x+yi) = (x^2 - y^2) + 2xyi : \mathbb{C} \to \mathbb{C}$ is continuous.
(this is just $z \mapsto z^2$)

(iii) $f(x+yi) = e^x\cos(y) + e^x\sin(y)i : \mathbb{C} \to \mathbb{C}$ continuous.

$(z \mapsto e^z)$

(iv) $f(x+yi) = \dfrac{1}{1-x^2-y^2} : D(0;1) \to \mathbb{R} \subset \mathbb{C}$ continuous

$\left(z \mapsto \dfrac{1}{1-|z|^2}\right)$ (does not extend to the boundary).

(7.5) Proposition. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a continuous function.

(i) For every open set $\mathcal{U} \subset \mathbb{C}$, $f^{-1}(\mathcal{U}) = \{z \in \Omega : f(z) \in \mathcal{U}\}$ is open.

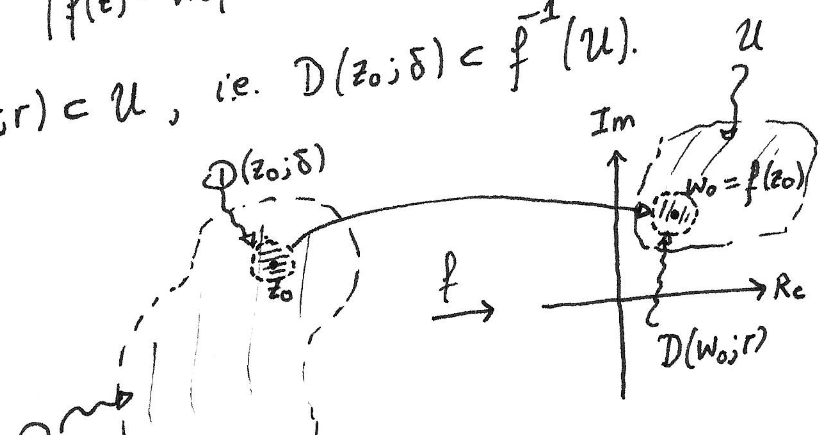
(ii) For every compact set K such that $K \subset \Omega$, $f(K) = \{f(z) : z \in K\}$ is compact.

<u>Proof of (i)</u> If $f^{-1}(\mathcal{U}) = \phi$, then there is nothing to prove. Assume $f^{-1}(\mathcal{U}) \neq \phi$ and let $z_0 \in f^{-1}(\mathcal{U})$. Let $w_0 = f(z_0) \in \mathcal{U}$. Since $\mathcal{U}$ is open, we have $r > 0$ such that $D(w_0;r) \subset \mathcal{U}$. Taking $\varepsilon = r$ in the definition of continuity (of f at z_0) we get a $\delta > 0$ such that $D(z_0;\delta) \subset \Omega$ (Ω is open)

and $|z - z_0| < \delta$ implies $|f(z) - w_0| < r$.

Meaning $f(D(z_0;\delta)) \subset D(w_0;r) \subset \mathcal{U}$, i.e. $D(z_0;\delta) \subset f^{-1}(\mathcal{U})$.

This proves that for every $z_0 \in f^{-1}(\mathcal{U})$ we can find $\delta > 0$ s.t. $D(z_0;\delta) \subset f^{-1}(\mathcal{U})$, i.e. $f^{-1}(\mathcal{U})$ is open.



$f(D(z_0;\delta)) \subset D(w_0;r)$

Remark.- (i) is infact equivalent to the definition of continuity: Meaning: let $\Omega \subset \mathbb{C}$ be an open set and $f: \Omega \to \mathbb{C}$. Then f is continuous $\Longleftrightarrow$ for every $\mathcal{U} \subset \mathbb{C}$ open set, $f^{-1}(\mathcal{U})$ is open.

(Easy exercise: prove this.)

<u>Proof of (ii)</u> One easy way to prove (ii) is by using Heine-Borel Theorem - a set is compact $\Longleftrightarrow$ every open cover has a finite subcover. Let K be a compact set, $K \subset \Omega$ as in the statement of (ii). Let $\{\mathcal{U}_j\}_{j\in J}$ be an open cover of $f(K)$. Then $\{f^{-1}(\mathcal{U}_j)\}_{j\in J}$ is an open cover of K ($f^{-1}(\mathcal{U}_j)$ is open by (i)). As K is compact, we can find finitely many $\mathcal{U}_{j_1}, \ldots, \mathcal{U}_{j_N}$ such that

$$K \subset \bigcup_{k=1}^{N} f^{-1}(\mathcal{U}_{j_k}).$$

This means

$$f(K) \subset \bigcup_{k=1}^{N} f(f^{-1}(\mathcal{U}_{j_k}))$$

Since $f(f^{-1}(X)) = X$ for any set X, we get $f(K) \subset \bigcup_{k=1}^{N} \mathcal{U}_{j_k}$, hence $f(K)$ is compact.

(7.6) Some consequences of Prop (7.5) (ii). - Next time

Analogue of extreme value theorem. $f: \Omega \to \mathbb{C}$ continuous K compact set; $K \subset \Omega$. Then $\exists\ \alpha, \beta \in K$ s.t. $|f(\alpha)| \leq |f(z)| \leq |f(\beta)|$ $\forall\ z \in K$.