Lecture 8

(8.0) Recall - last time we defined continuous functions. Given a set $A \subset \mathbb{C}$ and a function $f : A \to \mathbb{C}$, we say that <u>f is continuous</u> if for every $z_0 \in A$ and $\varepsilon > 0$, we can find $\delta > 0$ such that

$$|z - z_0| < \delta \text{ and } z \in A \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

In other words, $f(D(z_0;\delta) \cap A) \subset D(f(z_0);\varepsilon)$. We proved two important properties of continuous functions (Prop (7.5) - p5 of Lecture 7):

- inverse image of an open set is open.
- image of a compact set is compact.

(8.1) Distance from a compact set. Let $K \subset \mathbb{C}$ be a compact set and $z_0 \in \mathbb{C}$. Define

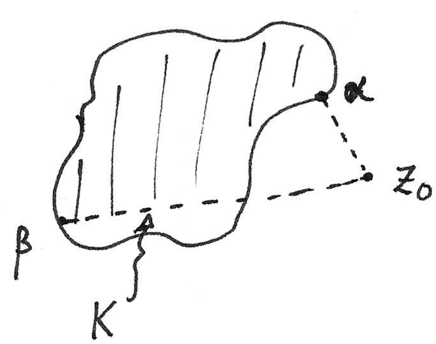
$$d_{z_0} : \mathbb{C} \to \cancel{\mathbb{C}}\, \mathbb{R} \subset \mathbb{C}. \qquad z \longmapsto |z - z_0|$$

This is a continuous function, hence $d_{z_0}(K) \subset \mathbb{R}_{\geq 0}$ is again compact, i.e., closed and bounded. This implies that there exist $\alpha, \beta \in K$ such that

$$d_{z_0}(\alpha) \leq d_{z_0}(z) \leq d_{z_0}(\beta) \quad \forall z \in K.$$

$$\mathrm{Dist}(z_0, K) := d_{z_0}(\alpha) = |z_0 - \alpha|$$

Since $\alpha \in K$, we have $D(z_0;K) > 0$ if and only if $z_0 \notin K$.



Remark.– We can define $\mathrm{Dist}(z_0; A) := \inf\{|z_0 - z| : z \in A\}$.

If A is not compact, it is possible to get $\mathrm{Dist}(z_0; A) = 0$, even when $z_0 \notin A$. For instance, $A = D(0;1)$ (open disc) and $z_0 = 1$ (an accumulation point not in A).

(8.2) Limit at ∞. For many applications, it is useful to know the behaviour of a function as its input "goes to infinity".

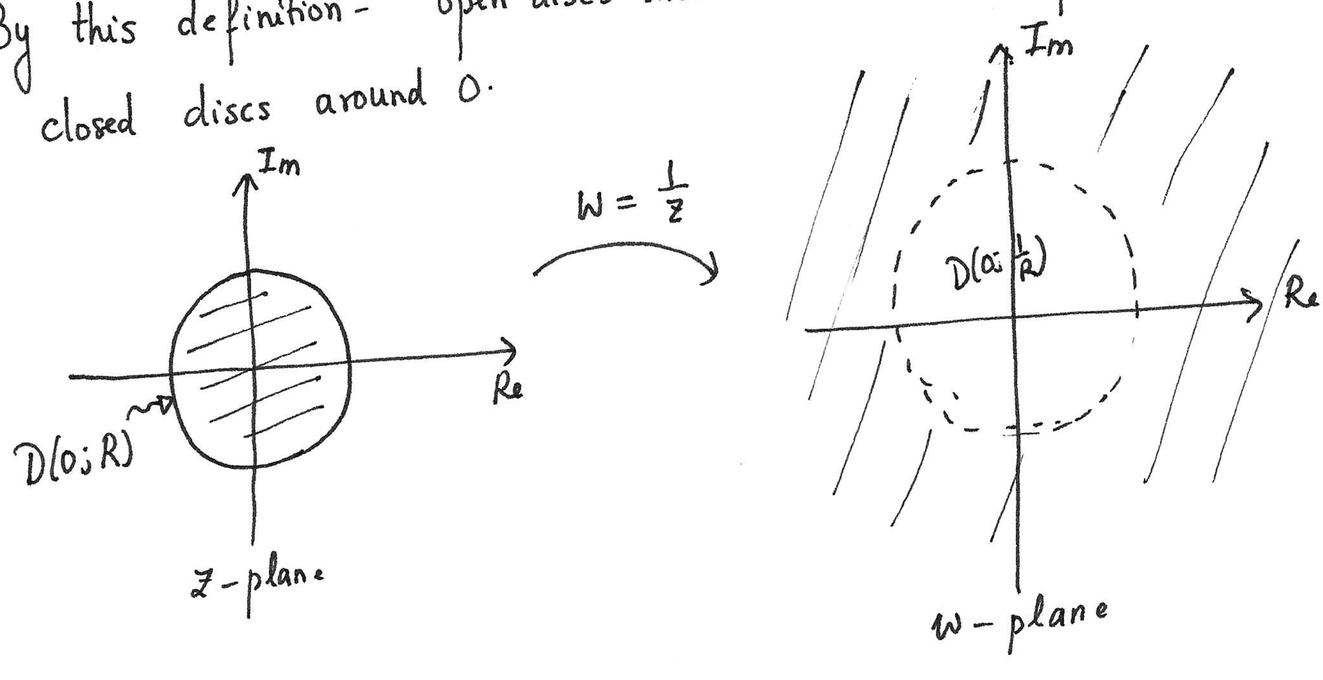
$\lim\limits_{\substack{z \to \infty \\ (z \in A)}} f(z)$ is only meaningful when A is an unbounded subset of $\mathbb{C}$.

In this case $\lim\limits_{z \to \infty} f(z) = \ell$ means that for every $\varepsilon > 0$ we can find $R > 0$ such that $\underset{(z \in A)}{|z| > R} \Rightarrow |f(z) - \ell| < \varepsilon$.

Note: this definition is equivalent to saying that $\lim\limits_{w \to 0} f\left(\frac{1}{w}\right) = \ell$.

e.g. $$\lim_{z \to \infty} \frac{z+1}{5z-7} = \lim_{z \to \infty} \frac{1 + \frac{1}{z}}{5 - \frac{7}{z}} = \lim_{w \to 0} \frac{1+w}{5-7w} = \frac{1}{5}.$$

By this definition - "open discs around ∞" are complements of closed discs around 0.

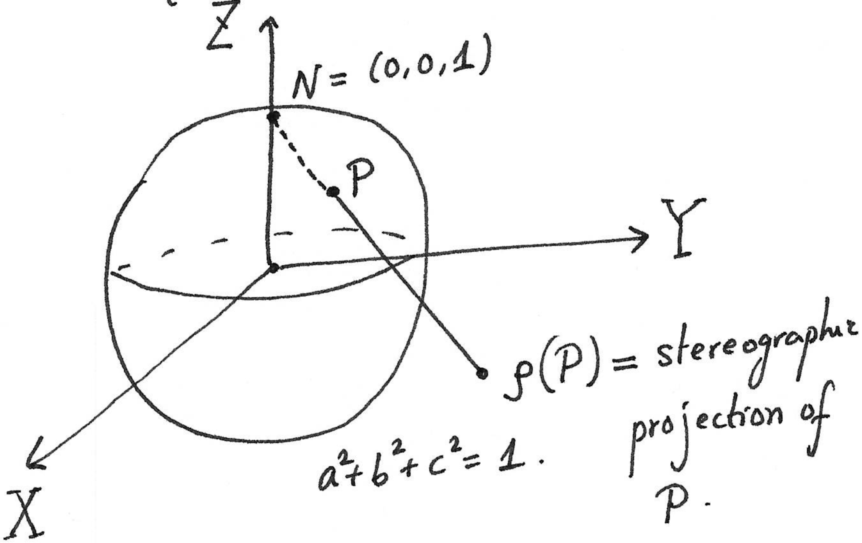


To understand "point at ∞" more concretely, Riemann defined stereographic projection.

(8.3) Stereographic projection. Consider the sphere of radius 1, centered at origin in $\mathbb{R}^3$: $\mathcal{S} = \{(a,b,c) : a^2+b^2+c^2 = 1\}$.

Stereographic projection is a function from $\mathcal{S} \setminus \{N\}$ to the X-Y plane, defined as follows.

north pole $(0,0,1)$



$\rho : \mathcal{S} \setminus \{N\} \longrightarrow \mathbb{R}^2$ (X-Y plane)

$\rho(P)$ = intersection point of X-Y plane and the line joining N & P.

Formula for ρ: If $P = (a,b,c)$ is a point on $\mathcal{S}$ ($c \neq 1$ so that $P \neq N$),

$\mathcal{L}$ = line joining N and P = $\{(ta, tb, 1+t(c-1)) : t \in \mathbb{R}\}$

$\mathcal{L} \cap (\text{X-Y plane}) = (ta, tb)$ where $t \in \mathbb{R}$ is such that $1+t(c-1) = 0$

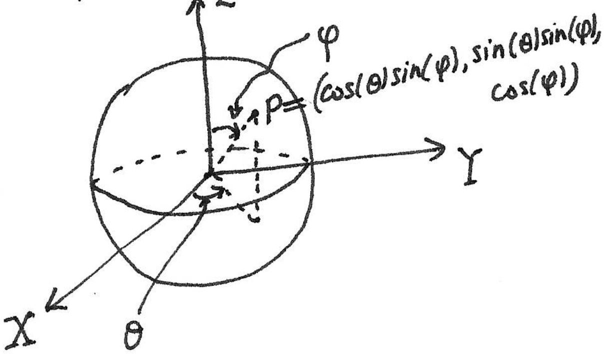
i.e. $t = \dfrac{1}{1-c}$. We get

$$\rho((a,b,c)) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

<u>Remark</u>. This map ρ identifies $\mathbb{R}^2$ viewed as a complex plane with sphere punctured at <u>north pole</u> N = "point at infinity"

It is clear that ρ maps upper hemisphere to the complement of the unit disc $D(0;1)$ and lower hemisphere to the unit disc. The circle $|z| = 1$ gets mapped to itself = equator.

(8.4) Stereographic projection in spherical coordinates. The sphere S is usually parametrized as $(\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$
$(-\pi < \theta \le \pi ; \ 0 \le \varphi \le \pi)$



If $P \in S$ has spherical coordinates θ and φ

then $$\rho(P) = \left(\cos(\theta)\,\frac{\sin(\varphi)}{1-\cos(\varphi)}, \ \sin(\theta)\,\frac{\sin\varphi}{1-\cos\varphi}\right)$$

Note $$\frac{\sin(\varphi)}{1-\cos(\varphi)} = \frac{2.\sin(\frac{\varphi}{2})\cos(\frac{\varphi}{2})}{2\sin^2(\frac{\varphi}{2})} = \cot(\tfrac{\varphi}{2})$$

i.e. $$\rho(P) = \cot(\tfrac{\varphi}{2})\,(\cos(\theta) + i\sin(\theta)) = \cot(\tfrac{\varphi}{2}).\,e^{i\theta}.$$
(switching to $\mathbb{C}$)

$$\left[\begin{array}{l} 0 \le \frac{\varphi}{2} \le \frac{\pi}{2} \\ \cot(0) = +\infty \leftarrow \rho \text{ is not defined} \\ \cot(\frac{\pi}{2}) = 0 \end{array}\right]$$

This formula makes it clear that any horizontal circle (φ = constant) on S is mapped to the circle, centered at 0, of radius $\cot(\frac{\varphi}{2})$.

(8.5) <u>Complex differentiability</u>. Let $\Omega \subset \mathbb{C}$ be an open set, $f: \Omega \to \mathbb{C}$ be a function. We say f is <u>$\mathbb{C}$-differentiable</u> at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists (denoted by } f'(z_0)).$$

$(h \in \mathbb{C}) \leftarrow$ this is the point where we depart from "real versions".

Note: as Ω is open, there exists $r>0$ s.t. $D(z_0;r) \subset \Omega$.

$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ signifies that $h \in \mathbb{C}$ is taken to be "small enough" so that $z_0+h \in D(z_0;r) \subset \Omega$ (i.e. $|h|<r$).

We say $f:\Omega \to \mathbb{C}$ is <u>$\mathbb{C}$-differentiable</u> if $f'(z_0)$ exists $\forall z_0 \in \Omega$.

(8.6) Some remarks. - (1) As usual <u>differentiability implies continuity</u>. That is, if $\Omega \subset \mathbb{C}$ is open, $f:\Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then f is continuous. To prove this, let $z_0 \in \Omega$ and $\varepsilon>0$ be given.

By definition, there exists $r>0$ s.t. $|h|<r$ implies

$$\left| \frac{f(z_0+h)-f(z_0)}{h} - f'(z_0) \right| < 1.$$

$\left[\text{of } \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)\right]$

$$\Rightarrow \quad |f(z_0+h)-f(z_0)| - |h \cdot f'(z_0)| \le |f(z_0+h)-f(z_0)-hf'(z_0)| \quad \left(\text{triangle ineq.}\right)$$
$$< |h|$$

$$\Rightarrow \quad |f(z_0+h)-f(z_0)| < |h|(1+|f'(z_0)|)$$ ← we want this to be $<\varepsilon$.

So, take $\delta < \operatorname{Min}\left\{ r, \frac{\varepsilon}{1+|f'(z_0)|} \right\}$. Then $|z-z_0|<\delta$ implies ($z = z_0+h$)

$|f(z)-f(z_0)| < \varepsilon$, proving that f is continuous at z_0 □

(2) Later we will change our terminology to reflect important consequences of $\mathbb{C}$-differentiability - which is a very strong notion. For instance,

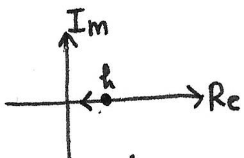
We will prove that f is $\mathbb{C}$-differentiable $\Rightarrow$ f' is $\mathbb{C}$-differentiable hence f is infinitely differentiable. When we prove this result (next week) we will start calling $\mathbb{C}$-differentiable functions "holomorphic".

(8.7) Cauchy-Riemann equations. Again let $f : \Omega \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function. Write $f(x+yi) = u(x,y) + i\,v(x,y)$.

Since the limit $\displaystyle\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists, we can let $h \to 0$ in whichever manner we please. (let $z_0 = x_0 + i y_0$)

(1) $h = t \in \mathbb{R}$.

$$f'(z_0) = \lim_{\substack{t \to 0 \\ (t \in \mathbb{R})}} \frac{u(x_0+t, y_0) - u(x_0, y_0) + i\left(v(x_0+t, y_0) - v(x_0, y_0)\right)}{t}$$

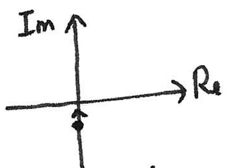


(h approaches 0 along real axis)

$$= u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

(2) $h = it$, $t \in \mathbb{R}$.

$$f'(z_0) = \lim_{\substack{t \to 0 \\ (t \in \mathbb{R})}} \frac{u(x_0, y_0+t) - u(x_0, y_0) + i\left(v(x_0, y_0+t) - v(x_0, y_0)\right)}{it}$$



h approaches 0 along Im-axis.

$$= \frac{1}{i}\left(u_y(x_0, y_0) + i\, v_y(x_0, y_0)\right) = v_y(x_0, y_0) - i\, u_y(x_0, y_0).$$

Thus we conclude:

$f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable, $f = u + iv$ $\Rightarrow$

$$\boxed{\begin{array}{l} u, v : \Omega \to \mathbb{R} \text{ satisfy} \\ \left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \text{Cauchy-Riemann equations} \end{array}}$$