Lecture 9

(9.0) Recall - last time we introduced the notion of <u>$\mathbb{C}$-differentiability</u>. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ a function. We say <u>f is $\mathbb{C}$-differentiable</u> if for every $z_0 \in \Omega$, the following limit exists

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad (\text{denoted by } f'(z_0)).$$

Writing $f(x+yi) = u(x,y) + i\,v(x,y)$, we showed that

$$f \text{ is } \mathbb{C}\text{-differentiable} \Rightarrow f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)$$
$$= v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

In particular u and v are related by the following system of partial differential equations:

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{ Cauchy-Riemann equations}$$

(9.1) Theorem. $[\Omega \subset \mathbb{C}$ open set$]$ $u, v : \Omega \to \mathbb{R}$ two (real) differentiable functions with continuous first derivatives (i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist at all points of Ω; and $u_x = \frac{\partial u}{\partial x}, u_y, v_x, v_y : \Omega \to \mathbb{R}$ are continuous).

If Cauchy-Riemann equations hold for u and v then $f = u + iv : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable. In this case $f' = u_x + i v_x = v_y - i u_y$.

Proof. The proof given below uses Taylor's theorem (special case - linear approximation for functions ($\mathbb{R}$-valued) of 2 real variables); stated below:

[Let $g : \Omega \to \mathbb{R}$ be such that g_x, g_y exist and are continuous, Consider $(x_0, y_0) \in \Omega$ and define

$$R_g(a,b) = g(x_0+a, y_0+b) - g(x_0,y_0) - a\, g_x(x_0,y_0) - b\, g_y(x_0,y_0)$$

[More precisely if $r>0$ is such that $\{(x-x_0)^2+(y-y_0)^2<r^2\} \subset \Omega$; $R_g(a,b)$ is defined on $\{a^2+b^2<r^2\}$.]

Then $$\lim_{(a,b)\to(0,0)} \frac{R_g(a,b)}{\sqrt{a^2+b^2}} = 0.$$

Let us continue with the proof of the theorem. Let $z_0 = x_0 + i y_0 \in \Omega$ and $h = a+bi$. We have to show that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = u_x(x_0,y_0) + i v_x(x_0,y_0) = v_y(x_0,y_0) - i u_y(x_0,y_0)$$

(write $\alpha = u_x(x_0,y_0) = v_y(x_0,y_0)$
$\beta = -u_y(x_0,y_0) = v_x(x_0,y_0)$)

$$\frac{f(z_0+h)-f(z_0)}{h} - (\alpha+\beta i) = \frac{\Big[\big(u(x_0+a,y_0+b)-u(x_0,y_0)\big) + i\big(v(x_0+a,y_0+b)-v(x_0,y_0)\big) - (a+bi)(\alpha+\beta i)\Big]}{h}$$

$$= \frac{u(x_0+a,y_0+b)-u(x_0,y_0) - a\alpha + b\beta}{h} + i\,\frac{v(x_0+a,y_0+b)-v(x_0,y_0) - a\beta - b\alpha}{h}$$

1st term: $$\frac{u(x_0+a,y_0+b)-u(x_0,y_0) - a u_x(x_0,y_0) - b u_y(x_0,y_0)}{h} = \frac{R_u(a,b)}{h}$$ (in the notation of Taylor's thm stated above)

2nd term: $$\frac{v(x_0+a,y_0+b)-v(x_0,y_0) - a v_x(x_0,y_0) - b v_y(x_0,y_0)}{h} = \frac{R_v(a,b)}{h}$$

As $\left|\frac{R_u(a,b)}{h}\right|$ and $\left|\frac{R_v(a,b)}{h}\right| \to 0$; as $|h|\to 0$ (by Taylor's theorem), we get that $\lim_{h \to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = \alpha + \beta i$ as claimed $\square$

(9.2) Theorem (9.1) gives a very practical way to determine $\mathbb{C}$-differentiability and compute $f'(z_0)$.

Examples. (i) $f(x+yi) = x - yi$ (i.e. $z \mapsto \bar{z}$). $\begin{aligned} u(x,y) &= x \\ v(x,y) &= -y. \end{aligned}$

Then $u_x = 1$, $u_y = 0$, $v_x = \overset{0}{\cancel{1}}$, $v_y = -1$. Cauchy-Riemann equations do not hold. So, f is not $\mathbb{C}$-differentiable.

(ii) $f(x+yi) = (x^2-y^2) + 2xyi$ (i.e. $z \mapsto z^2$).

$$\begin{aligned} u &= x^2 - y^2 \Rightarrow u_x = 2x \,;\; u_y = -2y \\ v &= 2xy \;\;\;\;\;\; \Rightarrow v_x = 2y \,;\; v_y = 2x \end{aligned} \qquad \text{So} \quad \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \quad \text{are also continuous,}$$

hence $f(z) = z^2$ is $\mathbb{C}$-differentiable, and $f'(z_0) = 2x_0 + 2y_0 i = 2z_0$.

Remark. - Later in the course we will prove that continuity of f' (i.e. u_x, v_x, u_y, v_y) is also a consequence of $\mathbb{C}$-differentiability.

(9.3) Alternate form of Cauchy-Riemann equations. Using the change of variables

$$\begin{aligned} z &= x + yi \\ \bar{z} &= x - yi \end{aligned} \qquad\qquad \begin{aligned} x &= \frac{z+\bar{z}}{2} \\ y &= \frac{z-\bar{z}}{2i} \end{aligned}$$

a given function $f : \Omega \to \mathbb{C}$ can be expressed either in variables x, y ; or $z, \bar{z}$. It is often inconvenient to re-express a function given in $z, \bar{z}$ to (x,y) variables and check Cauchy-Riemann equations. It is just more economical to obtain these equations in $z, \bar{z}$ directly.

Proposition. Cauchy-Riemann equations are equivalent to $\boxed{\dfrac{\partial f}{\partial \bar{z}} = 0}$.

Meaning. - if a function is expressed in $z, \bar{z}$ variables, for it to be $\mathbb{C}$-differentiable, it must not depend on $\bar{z}$ - hence "genuinely" function of one complex variable z.

Proof of Prop. : Note $\dfrac{\partial x}{\partial \bar{z}} = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial \bar{z}} = \dfrac{-1}{2i} = \dfrac{i}{2}$. Hence

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(u_x + i v_x\right) + \frac{i}{2}\left(u_y + i v_y\right)$$

$$= \frac{1}{2}\left(u_x - v_y\right) + \frac{i}{2}\left(u_y + v_x\right)$$

$$= 0 \iff \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}, \text{ Cauchy-Riemann equations} \quad \square$$

Example. $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ (polynomial in z) is $\mathbb{C}$-differentiable since $\dfrac{\partial f}{\partial \bar{z}} = 0$. [Note : in order to use Thm 9.1 - we have to check u_x, u_y, v_x, v_y are continuous - but these are merely polynomials in x, y in this example.]

Notation : $\boxed{\dfrac{df}{dz} = f'}$ Since a $\mathbb{C}$-differentiable function only depends on z and not $\bar{z}$; we don't have to use $\dfrac{\partial}{\partial z}$. (partial derivative).

(9.4) Example $f(x+yi) = e^x\cos(y) + e^x\sin(y)i.$ $(z \mapsto e^z)$

$u = e^x\cos(y)$ $\qquad v = e^x\sin(y)$

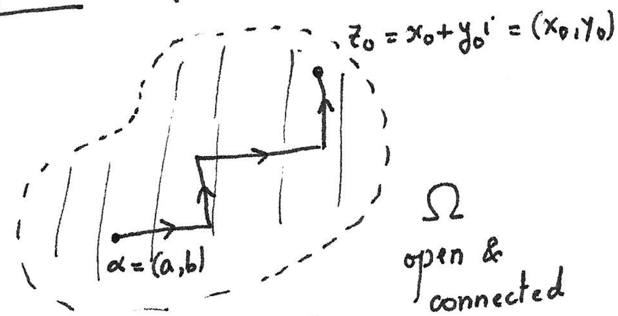
$u_x = e^x\cos(y)$ $\qquad v_x = e^x\sin(y)$

$u_y = -e^x\sin(y)$ $\qquad v_y = e^x\cos(y)$

- Cauchy-Riemann equations hold
- u_x, u_y are continuous.

So e^z is $\mathbb{C}$-differentiable, and
$$\boxed{\begin{aligned}\frac{d}{dz}e^z &= u_x + iv_x = e^x\cos(y) + e^x\sin(y)i \\ &= e^z\end{aligned}}$$

(9.5) Lemma. – Let $\Omega \subset \mathbb{C}$ be an open and connected set. Let $g : \Omega \to \mathbb{R}$ be a differentiable function s.t. $g_x = g_y = 0$ everywhere on Ω. Then g is a constant.



Proof. Let $\alpha \in \Omega$. Since for any $z_0 \in \Omega$ there exists a zig-zag path joining α to z_0, it is enough to show that the value of g remains constant along every horizontal & vertical line segment lying in Ω. We will only show the case of horizontal segments, the other one being entirely the same.

Write $\alpha = a + bi$. Then by Mean Value Theorem (applied to $x \mapsto g(x,b)$ - keeping b fixed)

we get that there exists $t \in (0,T)$ s.t.
$$g(a+T,b) - g(a,b) = g_x(a+t,b) = 0 \text{ by our assumption}$$
$$\Rightarrow g(a+T,b) = g(a,b)$$
$\square$

This lemma is used is determining whether a $\mathbb{C}$-differentiable function is constant.

(9.6) Examples. (i) $\underline{\Omega \subset \mathbb{C}\text{, open and connected}}$. Assume $f: \Omega \underset{(\mathbb{C}\text{-diff.})}{\to} \mathbb{C}$ is such that $Re(f) = c \in \mathbb{R}$ is constant. Then f is constant. (Pf.: $u_x = u_y = 0 \Rightarrow u_x = v_y = 0$; $-u_y = v_x = 0$ (Cauchy-Riemann eq.) hence v is also a constant).

(ii) If $f'(z_0) = 0 \ \forall z_0 \in \Omega$, then f is a constant.

(iii) Assume $f: \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable s.t. $|f(z)| = c$ is constant. Then f is a constant.

Proof. If $c = 0$, then $f(z) = 0 \ \forall z \in \Omega$, hence f is a constant. Assume $c \neq 0$ and write $u^2 + v^2 = \underset{\neq 0}{c^2}$. Taking derivatives, we get

$$\begin{aligned} u u_x + v v_x &= 0 \\ u u_y + v v_y &= 0 \end{aligned} \Rightarrow \begin{aligned} (1)\ u \cdot u_x - v \cdot u_y &= 0 \\ (2)\ u u_y + v u_x &= 0 \end{aligned} \quad \text{(Cauchy-Riemann)}$$

Solving for u_x and u_y, we get (multiply (1) by u and (2) by v - then add)

$$\begin{aligned} (u^2 + v^2) u_x &= 0 \\ (u^2 + v^2) u_y &= 0 \end{aligned} \Rightarrow \underset{(\text{since } u^2 + v^2 = c^2 \neq 0)}{u_x = u_y = 0}.$$

Hence $f' \equiv 0$ implying that f is constant. $\square$

(9.7) Harmonic functions. - Let $\Omega \subset \mathbb{C} = \mathbb{R}^2$ be an open set. $g: \Omega \to \mathbb{R}$ is called <u>harmonic</u> if it is twice differentiable (i.e. $g_{xx}, g_{xy} = g_{yx}, g_{yy}$ <u>exist and are continuous</u> - <u>notation $g \in C^2(\Omega)$</u>)

and $$\boxed{g_{xx} + g_{yy} = 0}$$ Laplace equation

Lemma. Let $f: \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable, $f = u + iv$.
Assume $u, v \in C^2(\Omega)$. Then u and v are harmonic.
[Later we will show that $u, v \in C^2(\Omega)$ follows from $\mathbb{C}$-differentiability of f].

Proof.

$$u_{xx} = v_{yx} \quad (\text{since } u_x = v_y) \qquad \text{and} \quad v_{yx} = v_{xy}$$

$$u_{yy} = -v_{xy} \quad (\text{since } u_y = -v_x) \qquad (\text{Clairaut's Theorem})$$

$$\Rightarrow u_{xx} + u_{yy} = 0. \quad (\text{Similarly } v_{xx} + v_{yy} = 0) \qquad \square$$

Remark.- Given $u \in C^2(\Omega)$; in order for there to exist $f: \Omega \to \mathbb{C}$, $\mathbb{C}$-differentiable such that $\mathrm{Re}(f) = u$; it is <u>necessary</u> that <u>u is harmonic</u>. $v = \mathrm{Im}(f)$ is to <u>be found</u> from <u>partial differential equations</u>

$$\boxed{\begin{aligned} v_x &= -u_y \\ v_y &= u_x \end{aligned}}$$

(often called <u>harmonic conjugate of u</u>). <u>Note</u>: This system may not have a solution on entire Ω, though "locally" it can be solved uniquely (up to a constant).

e.g. (i) $u(x,y) = x^2 - y^2$ is <u>harmonic</u>. Its harmonic conjugate is constrained to satisfy

$$v_x = -u_y = 2y \rightsquigarrow v = 2xy + \underbrace{C(y)}_{\text{constant depending on } y}$$

$$v_y = u_x = 2x$$

Set $v = 2xy + C(y)$ in the second eq$^{\underline{n}}$ to get $C'(y) = 0 \Rightarrow C \in \mathbb{R}$.

$\Rightarrow f(x+iy) = x^2 - y^2 + (2xy + C)i$: $\mathbb{C}$-diff. fn. s.t. $\mathrm{Re}(f) = u$.