Lecture 10

(10.0) Recall - given an open set $\Omega \subset \mathbb{C}$, a <u>harmonic function</u> on Ω is a (real-valued) $g : \Omega \to \mathbb{R}$ such that

(i) $g \in C^2(\Omega)$, i.e. $g_x, g_y, g_{xx}, g_{xy} = g_{yx}, g_{yy}$ exist and are continuous.

(ii) $$g_{xx} + g_{yy} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0 \quad \text{(Laplace* equation).}$$

$f = u + iv : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable (and $u, v \in C^2(\Omega)$) $\Rightarrow$ u and v are harmonic. [called harmonic conjugates of each other.]

Remarks. - (1) We will prove later in the course that $f = u + iv$ is $\mathbb{C}$-differentiable implies that u and v are infinitely differentiable.

(2) Given $u \in C^2(\Omega)$ solving Laplace equation, we get a system of partial differential equations (PDEs) for v - so that $u + iv$ is $\mathbb{C}$-diff.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

This system is consistent - however that only ensures existence of solutions near a point, not necessarily on entire domain Ω. We will have to impose some topological constraints on Ω (simply-connected-ness - later) that guarentee the existence of a solution $v : \Omega \to \mathbb{R}$ of these PDEs.

- See the example in the next section.

---

* Pierre-Simon Laplace (23/3/1749 - 5/3/1827)

(10.1)

~~(9.1)~~ Example. Let $\Omega = \mathbb{C} \setminus \{0\}$ and $u : \Omega \to \mathbb{R}$ be given by $u(x,y) = \frac{1}{2} \ln(x^2+y^2)$. Let us check that u is harmonic.

$$u_x = \frac{x}{x^2+y^2} . \quad u_{xx} = \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} .$$

As u is symmetric in x, y; we also get $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$.

$\Rightarrow u_{xx} + u_{yy} = 0$.

PDE's for the harmonic conjugate of u: $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y} = \dfrac{-y}{x^2+y^2}$.

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} .$$

Primitive (or antiderivative) with respect to y (keep x fixed) of the second equation $\dfrac{\partial v}{\partial y} = \dfrac{x}{x^2+y^2}$ is $v(x,y) = \arctan\left(\frac{y}{x}\right) + \underbrace{C(x)}_{\text{Constant depending on } x.}$

First eq^n $\dfrac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) = \dfrac{-y}{x^2+y^2} + C'(x) = \dfrac{-y}{x^2+y^2}$

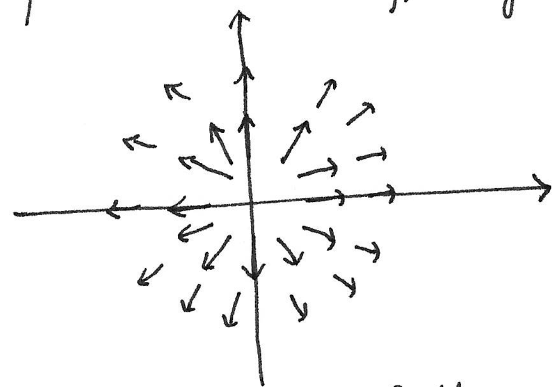
$\Rightarrow C(x) = C$ a constant. Thus $v : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ would be same as ~~a~~ "argument" $\arg(z)$ which, we know, is not continuous.

Remark. - The problem - as we will see later - is that $\mathbb{C} - \{0\}$ is NOT simply-connected. If we restrict our domain to $\Omega' = \mathbb{C} - \mathbb{R}_{\leq 0}$, $V(z) = \arg(z)$ is a genuine solution of PDE's written above and $u + iv = \log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \longrightarrow \mathbb{C}$, $\mathbb{C}$-differentiable.

(10.2) ~~(9.2)~~ Physical / geometric meaning of Cauchy-Riemann equations

[optional]

Historically, two real-valued functions of two real variables represented a vector field on $\mathbb{R}^2$. Physicists interpret vector fields differently - depending on the context.



(Picture of a vector field - given (x,y) we have placed a vector $\langle P(x,y), Q(x,y)\rangle$ at (x,y) - in this picture $P(x,y) = \dfrac{x}{x^2+y^2}$, $Q(x,y) = \dfrac{y}{x^2+y^2}$.)

- Clairaut* in 1740, observed that $\langle P(x,y), Q(x,y)\rangle$ = force vector $\vec{F}(x,y)$ acting on a particle placed at (x,y), then $\vec{F}$ is conservative (i.e. respects conservation of energy) if and only if $P_y = Q_x$.
- d'Alembert* in 1752, studying problems of hydrodynamics, imagined $\langle P(x,y), Q(x,y)\rangle$ as components of velocity of fluid flow (2D) at (x,y).

He observed that the flow is (explained below)

$$\left.\begin{array}{l} \text{irrotational} \longleftrightarrow P_y = Q_x \\ \text{incompressible} \longleftrightarrow P_x = -Q_y \end{array}\right\} \text{Cauchy-Riemann equations!}$$

d'Alembert also noticed that in this case $Q + iP$ can be viewed as a function of one variable $z = x + iy$.

(10.3) ~~(9.3)~~ [optional]

To understand the words irrotational / incompressible, it is necessary to review some key concepts of "line integrals of vector fields".

---

* Alexis Claude Clairaut (7/5/1713 - 17/5/1765); Jean Le Rond d'Alembert (17/11/1717 - 29/10/1783)

Given a vector field $\langle P(x,y), Q(x,y)\rangle = \vec{V}(x,y)$ in $\mathbb{R}^2$,

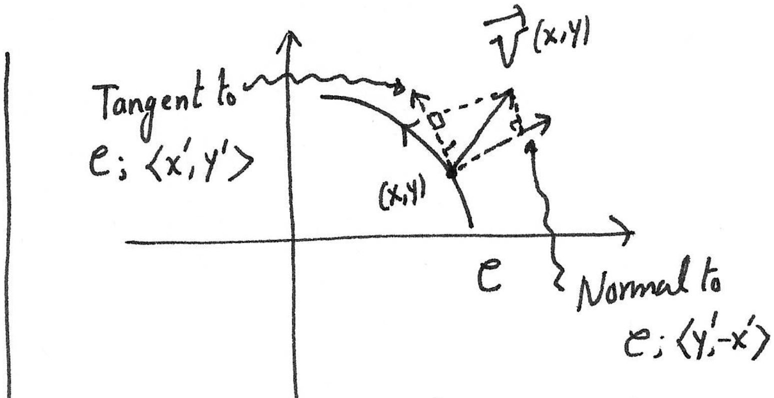
and a parametric curve $\mathcal{C} = (x(t), y(t))$ $(a \leq t \leq b)$ - we consider

2 quantities :

These integrals are independent of the parametri-zation of $\mathcal{C}$ (Though, depend on its orientation (± 1))

- flow along $\mathcal{C}$ $:= \displaystyle\int_a^b \Big(P(x(t),y(t))\, x'(t) + Q(x(t),y(t))\, y'(t)\Big)\, dt$
- flow across $\mathcal{C}$ $:= \displaystyle\int_a^b \Big(P(x(t),y(t))\, y'(t) - Q(x(t),y(t))\, x'(t)\Big)\, dt$

If $\mathcal{C}$ is closed, flow along $\mathcal{C}$ is also called <u>circulation</u> (or <u>curl</u>).; and flow across $\mathcal{C}$ is also called <u>flux</u>.



Flow along $\mathcal{C}$ = $\vec{V} \cdot \langle x'(t), y'(t)\rangle$ integrated over t

Irrotational $\leftrightarrow$ Circulation $= 0$ (along any closed $\mathcal{C}$)

Incompressible $\leftrightarrow$ Flux $= 0$

(10.4) ~~(9.4)~~ [optional] The development of "complex function theory" owes much to the discovery of equations of electromagnetism (Maxwell's* equations) in 1860's. I recommend Felix Klein's monograph - On Riemann's theory of algebraic functions - Introductory Chapter for more details. There, Klein views $u(x,y)$ as "velocity potential" - i.e. $\langle u_x, u_y\rangle$ = velocity vector. With this interpretation in mind.

Laplace equations $\longleftrightarrow$ Flow is steady

---

* James Clark Maxwell (13/6/1831 - 5/11/1879); Felix Klein (25/4/1849 - 22/6/1925)

(10.5) Properties of derivatives.

Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function. For $z_0 \in \Omega$, let

$$R_f(h) = f(z_0 + h) - f(z_0) - h f'(z_0).$$

Then, by definition $\lim_{h \to 0} \frac{R_f(h)}{h} = 0$. Using this, we easily obtain the following familiar properties of derivatives

(1) Linearity. – Let $f_1, f_2 : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable ($\Omega \subset \mathbb{C}$ open) and $\alpha_1, \alpha_2 \in \mathbb{C}$. Then

$$\frac{d}{dz}\left(\alpha_1 f_1(z) + \alpha_2 f_2(z)\right) = \alpha_1 f_1'(z) + \alpha_2 f_2'(z).$$

(2) Product (or Leibniz) rule

$$\left(f_1(z) f_2(z)\right)' = f_1'(z) f_2(z) + f_1(z) f_2'(z)$$

(3) Chain rule. Let $g : \Omega_2 \to \mathbb{C}$ and $f : \Omega_1 \to \mathbb{C}$ be $\mathbb{C}$-differentiable such that $g(\Omega_2) \subset \Omega_1$.

$$\begin{tikzcd} \Omega_2 \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \Omega_1 \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

Then $\frac{d}{dz}\left(f(g(z))\right) = f'(g(z)) \cdot g'(z)$

(4) L'Hôpital rule. Assume $f_1, f_2 : \Omega \to \mathbb{C}$ are $\mathbb{C}$-diff.; $z_0 \in \Omega$ be such that $f_1(z_0) = f_2(z_0) = 0$ and $f_2'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f_1(z)}{f_2(z)} = \frac{f_1'(z_0)}{f_2'(z_0)}$$

(10·6) Some examples.

(1) $\frac{d}{dz}(z) = 1$. Directly from the definition $\frac{z_0+h-z_0}{h} = 1$.

$$\Rightarrow \quad \frac{d}{dz}(z^n) = n \cdot z^{n-1}. \quad \text{(repeated application of Leibniz' rule)}$$

(This can also be proved directly from the definition - using binomial expansion: $\frac{(z_0+h)^n - z_0^n}{h} = \sum_{k=1}^{n} h^{k-1}\binom{n}{k} z_0^{n-k} \to n z_0^{n-1}$ as $h \to 0$.)

(2) We checked last time that $\frac{d}{dz}(e^z) = e^z$. Combined with chain rule, we get

$$\frac{d}{dz}(\sin(z)) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{i(e^{iz} + e^{-iz})}{2i}$$
$$= \cos(z).$$

Similarly, $\frac{d}{dz}(\cos(z)) = -\sin(z)$.

(3) Since $e^{\log(z)} = z$ we get $\frac{d}{dz}(\log(z)) = \frac{1}{z}$. (Ex.)

(4) $z^\alpha = e^{\alpha \cdot \log(z)} \Rightarrow \frac{d}{dz} z^\alpha = \alpha \cdot z^{\alpha - 1}$ (Ex.)

(10·7) Our next topic is integration of functions (over $\mathbb{C}$) along a parametric curve (or path) γ.

$$f : \Omega \to \mathbb{C} \quad \text{(continuous)} \quad \rightsquigarrow \quad \int_\gamma f(z)\,dz \in \mathbb{C}.$$
$$\gamma : [a,b] \to \Omega$$