Lecture 11

(11.0) Definition of line integrals. Let $\Omega \subset \mathbb{C}$ be an open set, $\gamma : [a,b] \to \Omega$ a path. We will assume that γ is <u>piecewise smooth</u> meaning there exist $c_1, c_2, \dots, c_n \in (a,b)$;

$$c_0 = a < c_1 < c_2 < \dots < c_n < b = c_{n+1}$$

such that $\gamma'(t)$ exists and is continuous $\forall\, t \in (c_j, c_{j+1})$; $j = 0,1,\dots,n$.

For a continuous function $f : \Omega \to \mathbb{C}$ define

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \quad \text{- for smooth } \gamma.$$

For piecewise smooth γ; $\displaystyle\int_\gamma f(z)\,dz = \sum_{j=0}^{n} \int_{c_j}^{c_{j+1}} f(\gamma(t))\,\gamma'(t)\,dt$; where

$c_0 = a < c_1 < \dots < c_n < b = c_{n+1}$ are as above.

(11.1) Real and imaginary parts. - If $f = u + iv$ and $\gamma(t) = x(t) + y(t)i$

then $$\int_\gamma f(z)\,dz = \int_a^b \Big(u(x(t),y(t)) + i\,v(x(t),y(t))\Big)\Big(x'(t) + i\,y'(t)\Big)\,dt$$

$$\Rightarrow \quad \mathrm{Re}\int_\gamma f(z)\,dz = \int_a^b \Big(u(x(t),y(t))\cdot x'(t) - v(x(t),y(t))\,y'(t)\Big)\,dt$$

$$\mathrm{Im}\int_\gamma f(z)\,dz = \int_a^b \Big(u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\Big)\,dt$$

(11.2) Properties of $\int_\gamma f(z)\,dz$. $\left[\underset{\text{open}}{\Omega \subset \mathbb{C}}\ ;\ f:\Omega \to \mathbb{C} \text{ continuous}\right]$

(1) Independence from parametrization. If $\gamma : [a,b] \to \Omega$ is a piecewise smooth path and $\tau : [c,d] \to [a,b]$ is differentiable on (c,d) with continuous derivative, we can consider $\mu : [c,d] \to \Omega$ given by $\mu(t) = \gamma(\tau(t))$.

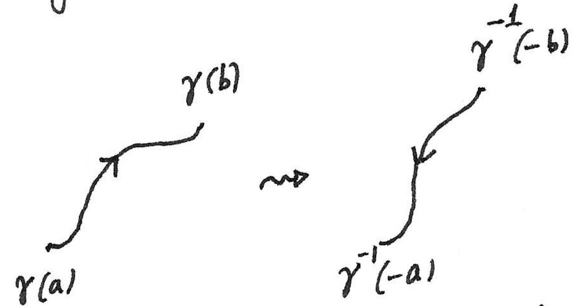
Then $$\int_\mu f(z)\,dz = \int_\gamma f(z)\,dz$$ (we are assuming $\tau(c) = a$ and $\tau(d) = b$)

Proof. $$\int_\mu f(z)\,dz = \int_c^d f(\mu(t))\,\mu'(t)\,dt = \int_c^d f(\gamma(\tau(t)))\,\gamma'(\tau(t))\,\tau'(t)\,dt$$

Set $\tau(t) = s$ to get $$\int_{\tau(c)=a}^{\tau(d)=b} f(\gamma(s))\,\gamma'(s)\,ds = \int_\gamma f(z)\,dz \qquad \square$$

(2) Note: $\int_\gamma f(z)\,dz$ depends on the direction of γ. If $\gamma : [a,b] \to \Omega$ we can define $\gamma^{-1} : [-b,-a] \to \Omega$ by $\gamma^{-1}(s) = \gamma(-s)$ $(-b \leq s \leq -a)$

(opposite or inverse of γ)



Then

$$\int_{\gamma^{-1}} f(z)\,dz = -\int_\gamma f(z)\,dz$$

(reversing the direction of γ).

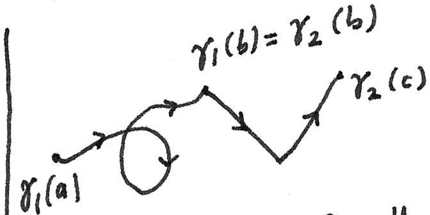
(3) Let $f_1, f_2 : \Omega \to \mathbb{C}$ be two continuous functions; $\alpha_1, \alpha_2 \in \mathbb{C}$.

Then
$$\int_\gamma \left(\alpha_1 f_1(z) + \alpha_2 f_2(z)\right) dz = \alpha_1 \int_\gamma f_1(z)\,dz + \alpha_2 \int_\gamma f_2(z)\,dz.$$

(4) <u>Concatenation of paths.</u> - Let $\gamma_1 : [a,b] \to \Omega$ and $\gamma_2 : [b,c] \to \Omega$ be two (piecewise smooth) paths. Define $\gamma = \gamma_1 \cdot \gamma_2 : [a,c] \to \Omega$

$$\gamma(t) = \begin{cases} \gamma_1(t) ; & a \le t \le b \\ \gamma_2(t) ; & b \le t \le c \end{cases}$$ - concatenation of γ_1 and γ_2.

Then
$$\int_{\gamma = \gamma_1 \cdot \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$



concatenation of paths

(5) Assume that there exists a $\mathbb{C}$-differentiable $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for every $z \in \Omega$. In this case, we say that f <u>admits an antiderivative or primitive</u> on Ω. Then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, this integral only depends on the end points of γ and not on the path connecting them. This is analogous to the fundamental theorem of calculus.

Proof Let $f = u + iv : \Omega \to \mathbb{C}$ (continuous), $F = U + iV : \Omega \to \mathbb{C}$ ($\mathbb{C}$-differentiable)

and $F' = f$; meaning
$$u = U_x = V_y$$
$$v = -U_y = V_x$$
(Cauchy-Riemann eq^{n}-s for F)

By (11.1) above, $\operatorname{Re} \int_\gamma f(z)\,dz = \int_a^b \big(u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\big)\,dt$

$$= \int_a^b \Big(U_x(x(t),y(t))\,x'(t) + U_y(x(t),y(t))\,y'(t)\Big)\,dt$$

$$= \int_a^b \frac{d}{dt}\Big(U(x(t),y(t))\Big)\,dt = U(x(b),y(b)) - U(x(a),y(a))$$

(by the fundamental Thm. of Calculus)

Similarly $\operatorname{Im} \int_\gamma f(z)\,dz = V(x(b),y(b)) - V(x(a),y(a))$ $\square$

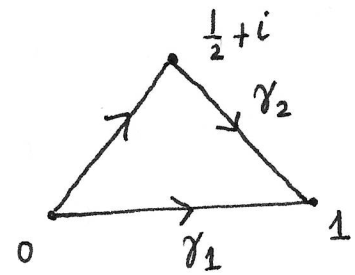
(11.3) Example I. Let $f(x+iy) = x$ (i.e. $f(z) = \operatorname{Re}(z)$).

Consider two paths joining 0 to 1 : $\gamma_1(t) = t \quad (0 \le t \le 1)$

$$\gamma_2(t) = \begin{cases} t + 2ti\,; & 0 \le t \le \frac{1}{2} \\ t + 2(1-t)i\,; & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\int_{\gamma_1} f(z)\,dz = \int_0^1 f(t)\cdot \overset{\gamma_1'(t)}{1}\cdot dt$$

$$= \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$$



$$\int_{\gamma_2} f(z)\,dz = \int_0^{1/2} f(t+2ti)\cdot \underbrace{(1+2i)}_{\gamma_2'(t)}\cdot dt + \int_{1/2}^1 f(t+(2-2t)i)\cdot \underbrace{(1-2i)}_{\gamma_2'(t)}\cdot dt$$

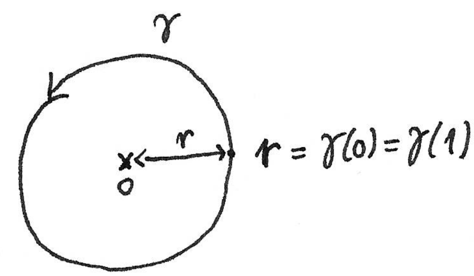
$$= (1+2i)\int_0^{1/2} t\,dt + (1-2i)\int_{1/2}^{1} t\cdot dt = (1+2i)\frac{1}{8} + (1-2i)\frac{3}{8}$$

$$= \frac{1}{2} - \frac{1}{2}i.$$

This example illustrates that $\int_\gamma f(z)\,dz$ is <u>not always</u> independent of the path joining $\gamma(a)$ to $\gamma(b)$. As we will see later, the problem with $f(z) = \mathrm{Re}(z)$ is that it is not $\mathbb{C}$-differentiable.

(11.4) Example II. $f(z) = \frac{1}{z} : \Omega \to \mathbb{C}$ (here $\Omega = \mathbb{C} - \{0\}$).

Let γ be counterclockwise oriented circle of radius $r > 0$, centered at 0. $\gamma(t) = r \cdot e^{it}$ $(0 \leq t \leq 2\pi)$



$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \underbrace{\frac{1}{r \cdot e^{it}}}_{f(\gamma(t))} \cdot \underbrace{r \cdot i \cdot e^{it}}_{\gamma'(t)}\,dt$$

$$= i \int_0^{2\pi} dt = 2\pi i.$$

If $\mu : [0,1] \to \Omega$ is the constant path connecting r to itself $(\mu(t) = r \;\; \forall\, 0 \leq t \leq 1)$, then $\int_\mu f(z)\,dz = 0$ (simply because $\mu'(t) = 0$)

Here, again, we see that $\int_\gamma f(z)\,dz \neq \int_\mu f(z)\,dz$ (the answer depends on the path joining r to itself) though $f(z) = \frac{1}{z}$ is $\mathbb{C}$-differentiable.

The problem here is that the region enclosed by γ, $D(0;r)$, is not in the domain of f $(D(0;r) \not\subset \Omega = \mathbb{C} - \{0\})$.

(11.5) Example III. $f(z) = z^3 : \Omega \to \mathbb{C}$ $(\Omega = \mathbb{C})$.

Let γ be any path joining 1 to $2+i$. Then (by property (5). - page 3 above)

$$\int_\gamma f(z)\,dz = \left[\frac{z^4}{4}\right]_1^{2+i} = \frac{(2+i)^4 - 1}{4}$$

$$= \frac{-8+24i}{4} = -2+6i.$$

(11.6) Theorem*. Let $\Omega \subset \mathbb{C}$ be an open and connected set, $f : \Omega \to \mathbb{C}$ be a continuous function. Then the following conditions are equivalent.

(1) f admits an antiderivative.

(2) $\int_\gamma f(z)\,dz$ depends only on the endpoints of γ - for every piecewise smooth path γ in Ω.

(3) $\int_\gamma f(z)\,dz$ depends only on the endpoints of γ - for every zig-zag path γ in Ω.

Proof. (1) $\Rightarrow$ (2) by Property (5) on page 3 above.

(2) $\Rightarrow$ (3) is clear since a zig-zag path is piecewise smooth.

(3) $\Rightarrow$ (1). The idea of the proof is exactly the same as for functions of a real variable. Namely given $g(x)$ we find its primitive by fixing a point x_0 on the real line and $G(x) := \int_{x_0}^{x} g(t)\,dt$. While over $\mathbb{R}$ there is only one way to go from x_0 to x; over $\mathbb{C}$ - there are many paths

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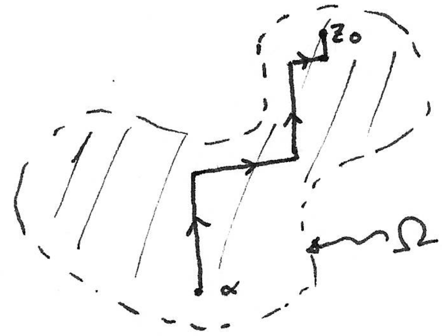
* This result is often attributed to Giacinto Morera (18/7/1856 – 8/2/1907)

joining two given points – and we need to make the assumption as in (3) to remove the dependence on the path chosen.

So, choose $\alpha \in \Omega$, and define $F : \Omega \to \mathbb{C}$ by $(z_0 \in \Omega)$

$$F(z_0) := \int_{\gamma} f(z)\,dz \quad ; \quad \gamma \text{ is any zig-zag path joining } \alpha \text{ to } z_0.$$



Such path exists since Ω is open & connected (see Prop. 6.5 – Lecture 6).

The definition of $F(z_0)$ is independent of the choice of γ (i.e. F is well-defined) by our assumption (3).

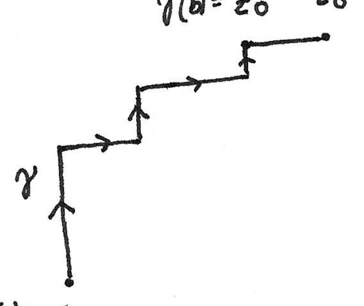
We have to prove that F is $\mathbb{C}$-differentiable, and $F' = f$. For this, it is enough to prove (by Thm 9.1 – Lecture 9 – applied to F) : $\left[\begin{array}{l} F = U + iV \\ f = u + iv \end{array} \text{ below}\right]$

(i) $U_x = V_y = u$ (ii) $-U_y = V_x = v$

(since u & v are assumed to be continuous – the hypotheses of Thm 9.1 for U and V will hold – once (i) & (ii) are proved).

Proof of $U_x = u$ & $V_x = v$. Let $z_0 \in \Omega$ and let $h \in \mathbb{R}$.

$\underbrace{\gamma_H : t \mapsto z_0 + th}$ $(0 \le t \le 1)$



Let γ be a zig-zag path joining α to z_0.

$\gamma_H(t) = z_0 + th$ horizontal segment joining z_0 to $z_0 + h$ $(0 \le t \le 1)$

Then $$F(z_0 + h) = \int_{\gamma \cdot \gamma_H} f(z)\,dz = \int_{\gamma} f(z)\,dz + \int_{\gamma_H} f(z)\,dz = F(z_0) + \int_{\gamma_H} f(z)\,dz$$

$$\Rightarrow F(z_0+h) - F(z_0) = \int_{\gamma_H} f(z)\,dz = \int_0^1 f(z_0+th)\cdot \underbrace{h}_{\gamma_H'(t)}\cdot dt$$

$$\Rightarrow \frac{F(z_0+h)-F(z_0)}{h} = \int_0^1 f(z_0+th)\,dt.$$

Claim: $\lim\limits_{h\to 0} \int_0^1 f(z_0+th)\,dt = f(z_0)$. This claim proves that

$\lim\limits_{\substack{h\to 0\\(h\in\mathbb{R})}} \frac{F(z_0+h)-F(z_0)}{h} = U_x(x_0,y_0) + i\,V_x(x_0,y_0)$ exists, and $U_x(x_0,y_0) = u(x_0,y_0)$, $V_x(x_0,y_0) = v(x_0,y_0)$.

The proof of $-U_y(x_0,y_0) = v$ & $V_y(x_0,y_0) = u$ is similar (replace γ_H (horizontal) by γ_V (vertical)) and is left as an easy check.

Proof of the claim: Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left| \int_0^1 f(z_0+th)\,dt - f(z_0) \right| < \varepsilon.$$

Since f is continuous, there is $\delta > 0$ so that $|f(z) - f(z_0)| < \varepsilon$ for every z s.t. $|z - z_0| < \delta$.

This δ works for us:

for $|h| < \delta$, we get $\left| \int_0^1 f(z_0+th)\,dt - f(z_0) \right| = \left| \int_0^1 \big(f(z_0+th) - f(z_0)\big)\,dt \right|$

$$\leq \int_0^1 |f(z_0+th) - f(z_0)|\,dt \quad \text{(triangle inequality)}$$

$$< \int_0^1 \varepsilon\cdot dt = \varepsilon. \quad \left(\begin{array}{l}\text{since } |h|<\delta \text{ and } 0\leq t\leq 1\\ \Rightarrow |th| < \delta\end{array}\right)$$

$\square$