Lecture 12

(12.0) Recall - last time we defined $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$

(here $\underset{\text{open}}{\Omega\subset\mathbb{C}}$; $\underset{\text{continuous}}{f:\Omega\to\mathbb{C}}$; $\gamma:[a,b]\to\Omega$ piecewise smooth path).

We showed that, for Ω open & connected, the following statements are equivalent: (1) f admits an antiderivative.

(2) $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$ for every $\gamma_1,\gamma_2 : [a,b]\to\Omega$ piecewise smooth s.t. $\gamma_1(a)=\gamma_2(a)$, $\gamma_1(b)=\gamma_2(b)$

(3) Same statement as (2) - for every (two) zig-zag paths γ_1 and γ_2.

(Thm. 11.6 - Lecture 11).

(12.1) An important inequality. for $f:\Omega\to\mathbb{C}$ continuous & $\gamma:[a,b]\to\mathbb{C}$ piecewise smooth,

$$\boxed{\left|\int_\gamma f(z)\,dz\right| \leq M\cdot L}$$

where $M = \mathrm{Max}\{|f(\gamma(t))| : a\leq t\leq b\}$; $L = \mathrm{Length}(\gamma) = \int_a^b |\gamma'(t)|\,dt$

Proof. We verify that, for any $g:[a,b]\to\mathbb{C}$ continuous,

$$\left|\int_a^b g(t)\,dt\right| \leq \int_a^b |g(t)|\,dt \quad (\text{for us; } g(t) = f(\gamma(t))\cdot\gamma'(t))$$

This is essentially triangle inequality - using Riemann Sums definition of definite integral.

Riemann Sums: Pick $N \geq 1$ and subdivide $[a,b]$ into N intervals

- $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ (e.g. $t_j = a + j \cdot \Delta$, where $\Delta = (b-a)/N$)
- Choose $\xi_j \in [t_j, t_{j+1}]$ $(0 \leq j \leq N-1)$. (e.g. we can take $\xi_j = t_j$)
- Take the finite sum $\sum_{j=0}^{N-1} g(\xi_j)(t_{j+1} - t_j)$

Then
$$\int_a^b g(t)\,dt = \lim_{N \to \infty} \sum_{j=0}^{N-1} g(\xi_j)(t_{j+1} - t_j).$$

Now, by triangle inequality,
$$\left| \sum_{j=0}^{N-1} g(\xi_j)(t_{j+1} - t_j) \right| \leq \sum_{j=0}^{N-1} |g(\xi_j)|(t_{j+1} - t_j)$$

which proves that
$$\left| \int_a^b g(t)\,dt \right| \leq \int_a^b |g(t)|\,dt.$$

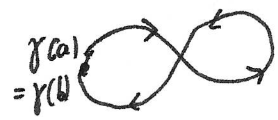
For $g(t) = f(\gamma(t)) \cdot \gamma'(t)$, we get:
$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \right|$$
$$\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,dt \leq M \int_a^b |\gamma'(t)|\,dt$$
$$= M \cdot L$$

$\square$

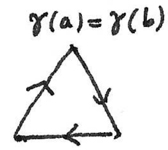
(12.2) Definition. A path $\gamma : [a,b] \to \mathbb{C}$ is said to be closed (or a loop) if $\gamma(a) = \gamma(b)$. We say γ is simple if it does not cross itself, except possibly at the end points. More precisely

(γ is simple):
$$a < t_1 < t_2 < b \Rightarrow \gamma(t_1) \neq \gamma(t_2)$$
$$a < t < b \Rightarrow \gamma(t) \neq \gamma(a) \ \& \ \gamma(t) \neq \gamma(b).$$

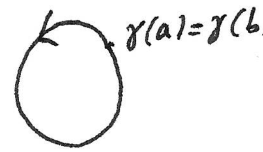
Neither simple nor closed



closed, not simple



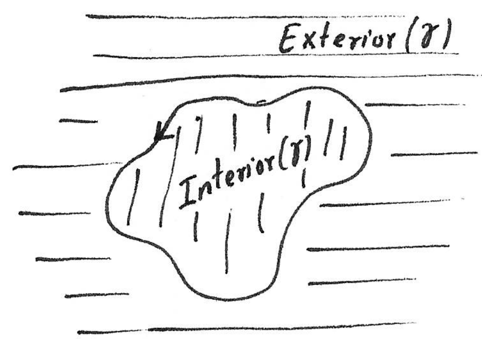
simple, closed



simple, closed.

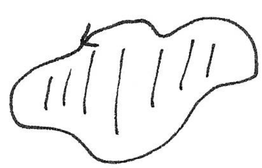
(12.3) Interior, exterior and orientation. Let γ be a simple, closed path in $\mathbb{C}$. Then, the complement $\mathbb{C} \setminus \gamma$ has exactly two connected components (Jordan* Curve Theorem).

Bounded Component = Interior of γ
Unbounded Component = Exterior of γ



We say γ is positively oriented (or counterclockwise) if Interior of γ is to the left ~~of~~ while traversing γ

e.g  clockwise

 counterclockwise

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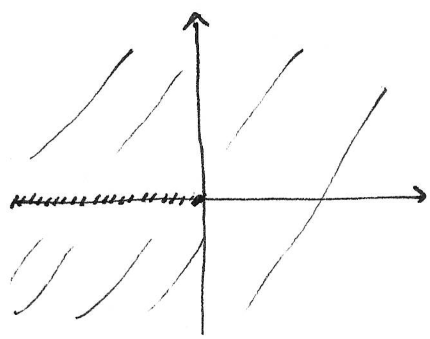
* Camille Jordan (5/1/1838 – 22/1/1922). We are not going to prove Jordan Curve Thm in this course.

(12.4) Simply-connected sets. A subset $A \subset \mathbb{C}$ is said to be <u>simply connected</u> if for every simple, closed path $\gamma : [a,b] \to A$, $\text{Interior}(\gamma) \subset A$.

e.g. $\mathbb{C}$ is simply-connected. Open disc $D(z_0;R)$ is simply-connected.

$\mathbb{C} \setminus \{0\}$, $\overset{\times}{D}(z_0;R)$ (punctured disc) are not simply-connected.

$\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is simply-connected



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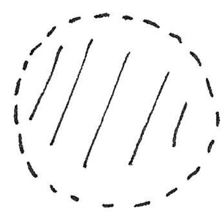

$D(z_0;R)$

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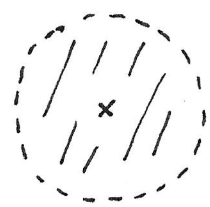


$D^{\times}(z_0;R)$

X

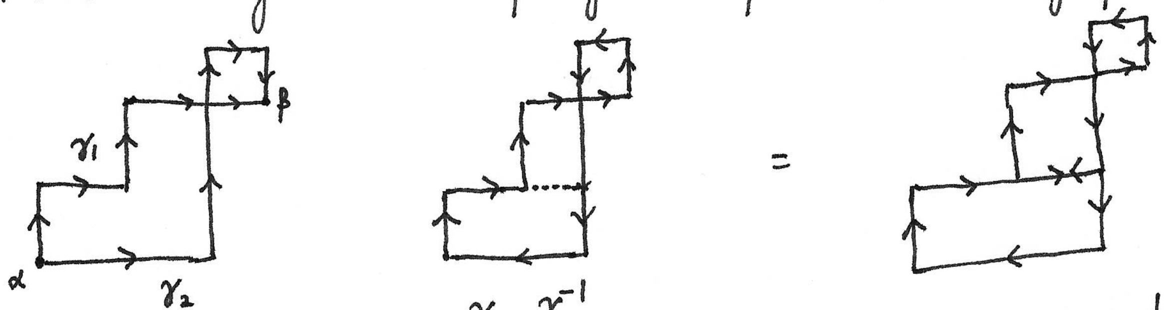
(12.5) Cauchy's Theorem. Let $\Omega \subset \mathbb{C}$ be an open, connected and <u>simply-connected</u> set. Let $f : \Omega \to \mathbb{C}$ be <u>$\mathbb{C}$-differentiable</u>. Then, for any <u>closed path</u> $\gamma : [a,b] \to \Omega$,

$$\boxed{\int_\gamma f(z)\,dz = 0}$$

Proof. Using Theorem (11.6) – it is enough to show that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for any two zig-zag paths γ_1 and γ_2 having the same endpoints. Since,

$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_1\gamma_2^{-1}} f(z)\,dz$, equivalently we have to

Show that $\int_\gamma f(z)\,dz = 0$ for any closed zig-zag path. Note that any closed zig-zag path is a union of rectangles. So, we have reduced the general case of any closed path to boundary of a rectangle.



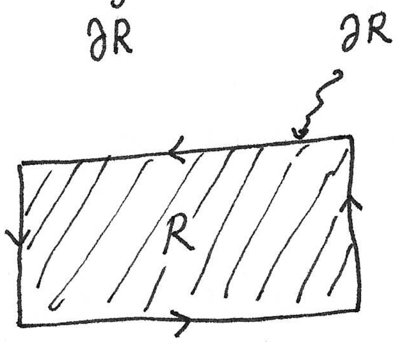
Two zig-zag paths joining α to β

$\gamma_1 \cdot \gamma_2^{-1}$ closed zig-zag path

sum over rectangular paths

<u>To prove</u>: Let R be a rectangle in Ω. Then $\int_{\partial R} f(z)\,dz = 0$

(∂R = boundary of R)



Let D = length of the longer diagonal of R

P = perimeter of R.

Given any $\varepsilon > 0$, we will eventually prove that

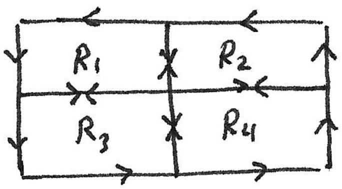
$$\left| \int_{\partial R} f(z)\,dz \right| < \varepsilon \cdot D \cdot P \; ; \text{ which implies}$$

that

$$\int_{\partial R} f(z)\,dz = 0.$$

Step 1. Divide R into 4 equal rectangles R_1, R_2, R_3, R_4.

Then $$\int_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial R_j} f(z)\,dz$$



Step 2. Let $k \in \{1,2,3,4\}$ be such that

$$\left| \int_{\partial R_k} f(z)\,dz \right| \geq \left| \int_{\partial R_j} f(z)\,dz \right| \quad \text{for each } j = 1,2,3,4.$$

Step 3. Let us call $R_k = R^{(1)}$. Then, by triangle inequality

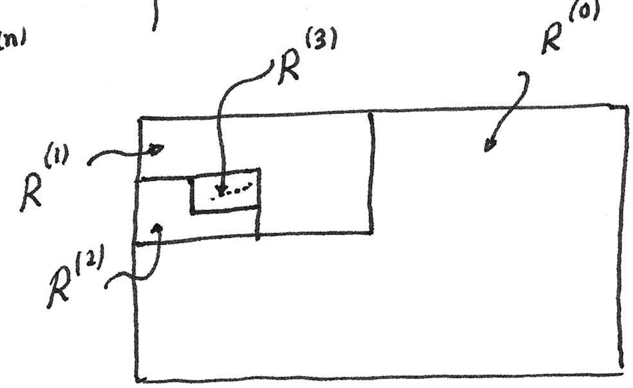
$$\left| \int_{\partial R} f(z)\,dz \right| \leq 4 \cdot \left| \int_{\partial R^{(1)}} f(z)\,dz \right|.$$

Also $D^{(1)} =$ diameter (length of the longer diagonal) of $R^{(1)} = \frac{1}{2} D$.

$p^{(1)} =$ perimeter of $R^{(1)} = \frac{1}{2} P$.

Repeating Steps 1 - 3 for $R^{(1)}$, we get $R^{(2)}$ and continuing this way, we obtain a nested sequence of shrinking rectangles $R^{(0)} = R \supset R^{(1)} \supset R^{(2)} \supset \cdots$

$$\left.\begin{aligned} D^{(n)} &= \text{diameter of } R^{(n)} = \frac{1}{2^n} D \\ p^{(n)} &= \text{perimeter of } R^{(n)} = \frac{1}{2^n} \cdot P \\ \left| \int_{\partial R} f(z)\,dz \right| &\leq 4^n \left| \int_{\partial R^{(n)}} f(z)\,dz \right| \end{aligned}\right\} \quad (*)$$



Now $$\bigcap_{n=0}^{\infty} R^{(n)} = \{z_0\}$$

(argue by taking real and imaginary parts: if $R^{(n)} = \left\{ \begin{matrix} a_n \leq \mathrm{Re}(z) \leq b_n \\ c_n \leq \mathrm{Im}(z) \leq d_n \end{matrix} \right\}$ then $\begin{matrix} a_0 \leq a_1 \leq a_2 \leq \cdots\cdots \leq b_2 \leq b_1 \leq b_0 \\ c_0 \leq c_1 \leq c_2 \leq \cdots\cdots \leq d_2 \leq d_1 \leq d_0 \end{matrix}$

moreover $\begin{array}{l} |b_n - a_n| \to 0 \\ |d_n - c_n| \to 0 \end{array}$ as $n \to \infty$. $\Rightarrow$ $\bigcap_{n=0}^{\infty} [a_n, b_n] = \{x_0\}$

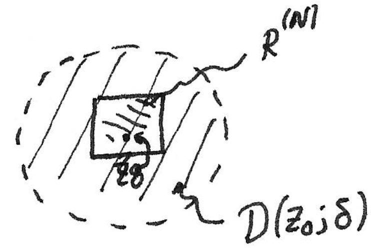
$$\bigcap_{n=0}^{\infty} [c_n, d_n] = \{y_0\}$$

Hence $\bigcap_{n=0}^{\infty} R^{(n)} = \{x_0 + i y_0\}$.)

Now, let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that

$$|z - z_0| < \delta \quad \Rightarrow \quad \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

(exists, since f is $\mathbb{C}$-differentiable). Pick $N \gg 0$ s.t. $D(z_0; \delta)$ contains $R^{(N)}$



Using $\int_{\partial R^{(N)}} f(z_0)\,dz = f(z_0) \int_{\partial R^{(N)}} 1 \cdot dz = 0$

and $\int_{\partial R^{(N)}} (z - z_0) f'(z_0)\,dz = f'(z_0) \int_{\partial R^{(N)}} (z - z_0)\,dz = 0$

($(z - z_0)$ has an antiderivative on $\mathbb{C}$)

we get $\int_{\partial R^{(N)}} f(z)\,dz = \int_{\partial R^{(N)}} \left(f(z) - f(z_0) - (z - z_0) f'(z_0) \right) dz$

$$\Rightarrow \left| \int_{\partial R^{(N)}} f(z)\,dz \right| \leq \int_{\partial R^{(N)}} \left| f(z) - f(z_0) - (z - z_0) f'(z_0) \right| dz < \int_{\partial R^{(N)}} \varepsilon |z - z_0|\,dz$$

$$\leq \varepsilon \cdot \mathrm{Max}\left\{ |z - z_0| : z \in \partial R^{(N)} \right\} \cdot \mathrm{Length}\left(\partial R^{(N)}\right)$$

$$\leq \varepsilon \cdot D^{(N)} \cdot P^{(N)}.$$

Hence, $\left| \int_{\partial R} f(z)\,dz \right| \leq 4^N \left| \int_{\partial R^{(N)}} f(z)\,dz \right| < \varepsilon \cdot 4^N \cdot \frac{1}{2^N} D \cdot \frac{1}{2^N} P = \varepsilon \cdot D \cdot P$

as we wanted to show (see page 5 above) $\square$

(12.6) Some remarks. (1) Augustin-Luis Cauchy (21/8/1789 – 23/5/1857) first published a proof of Thm (12.5) in 1842., under an addition hypothesis – that u_x, u_y, v_x, v_y are continuous. $\begin{pmatrix} u = \mathrm{Re} f \\ v = \mathrm{Im} f \end{pmatrix}$ His original proof used Green's theorem* which was proved in 1828.

Green's Theorem. – Let $P, Q : \Omega \to \mathbb{R}$ be in $C^1(\Omega)$, γ be a counterclockwise oriented, simple, closed curve bounding a region $A \subset \Omega$. Then

$$\int_\gamma P\,dx + Q\,dy = \iint_A (Q_x - P_y)\,dx\,dy$$

Cauchy's original proof – $\mathrm{Re} \int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy = \iint_A (-v_x - u_y)\,dx\,dy$

$= 0$ by Cauchy-Riemann eq^n's. Similarly for $\mathrm{Im} \int_\gamma f(z)\,dz$

(2) The more general statement given in (12.5) and its first proof were obtained by Eduard Goursat (21/5/1858 – 25/11/1936) in 1884. Sometimes Thm 12.5 is called Cauchy-Goursat theorem.

(3) Alternate formulation of Thm 12.5. Let $f : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable, $\Omega \subset \mathbb{C}$ open, connected. Let $\gamma : [a,b] \to \Omega$ be a simple closed curve such that $\mathrm{Interior}(\gamma) \subset \Omega$. Then $\int_\gamma f(z)\,dz = 0$.

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* Georg Green 14/7/1793 – 31/5/1841