Lecture 13

(13.0) Recall that in the last lecture we proved Cauchy's Theorem.

There are two ways to formulate it.

$\int_\gamma f(z)\,dz = 0$ under either of the following two assumptions on Ω and γ (here $f:\Omega\to\mathbb{C}$ is $\mathbb{C}$-differentiable). $\Omega\subset\mathbb{C}$ is open and connected.

(i) Ω is simply-connected and γ is a closed path (piecewise smooth) in Ω.

(ii) γ is simple, closed, piecewise smooth path in Ω and Interior $(\gamma)\subset\Omega$.

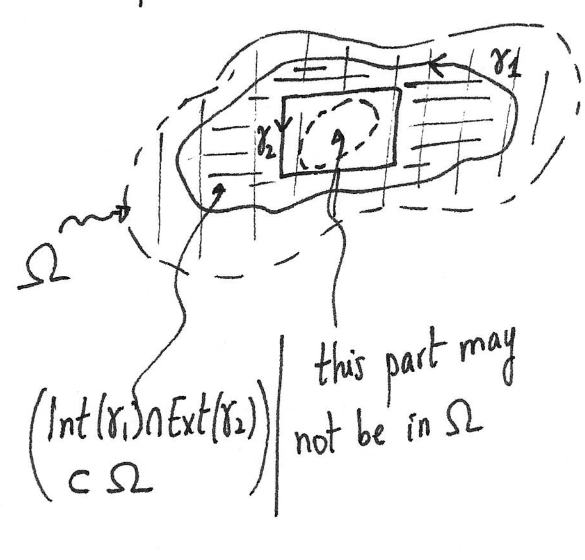
A simple, closed, piecewise smooth path is also called a contour. If nothing is specified, we will assume that a contour is counterclockwise oriented.

(13.1) First application of Cauchy's Theorem – principle of contour deformation.

Let $f:\Omega\to\mathbb{C}$ be $\mathbb{C}$-differentiable ($\Omega\subset\mathbb{C}$ open, connected).

Let γ_1 be a contour in Ω; γ_2 another contour in $\Omega\cap\text{Interior}(\gamma_1)$.

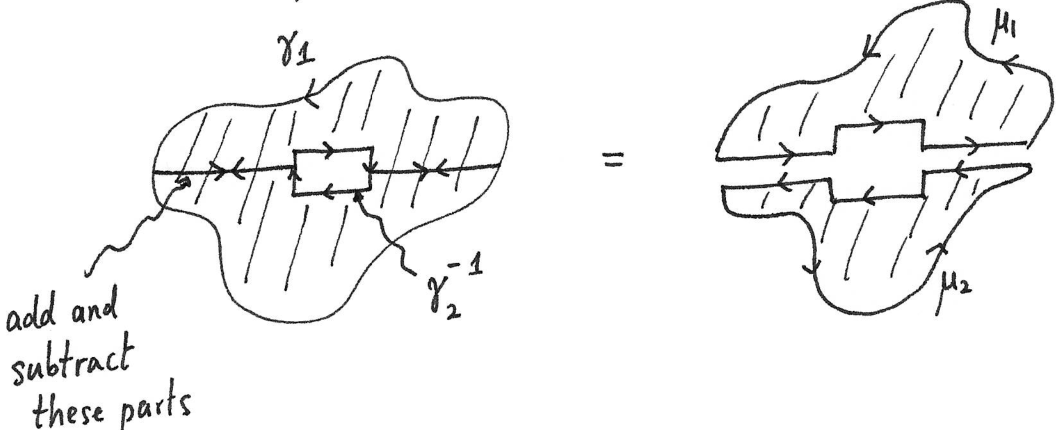
If $\text{Interior}(\gamma_1)\cap\text{Exterior}(\gamma_2)\subset\Omega$, then $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$



Remark.- This principle allows us to choose "convenient" contours in explicit calculations.

Idea of a proof.- We can write $\displaystyle\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$ as

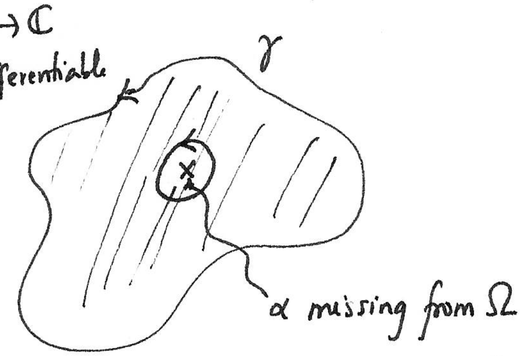
$\displaystyle\int_{\mu_1} f(z)\,dz + \int_{\mu_2} f(z)\,dz$; where μ_1 and μ_2 satisfy the hypothesis (ii) of (13.0)



$$\left(\int_{\mu_1} + \int_{\mu_2} = \int_{\gamma_1} - \int_{\gamma_2} \right)$$

$\square$

Special case: Let γ be a contour in Ω such that there is $\alpha \in \text{Interior}(\gamma)$ with $\text{Interior}(\gamma) - \{\alpha\} \subset \Omega$. $f:\Omega \to \mathbb{C}$ $\mathbb{C}$-differentiable



$\text{Int}(\gamma) - \{\alpha\} \subset \Omega.$

Then $$\int_{\gamma} f(z)\,dz = \int_{\mathcal{C}(\alpha;r)} f(z)\,dz$$

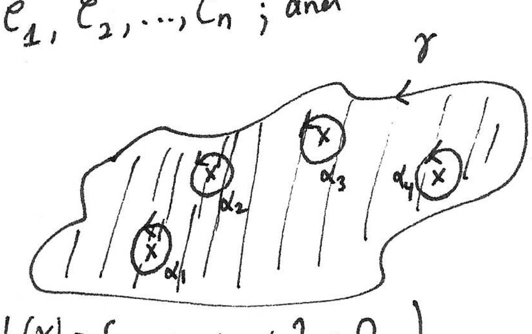
$\mathcal{C}(\alpha,r)$ = (counterclockwise) circle of radius $r>0$ centered at α.

[we can take $r>0$ to be as small as we like.]

→ Similarly for finite number of α's missing from Ω: i.e.

If $\alpha_1, \dots, \alpha_n \in \text{Int}(\gamma)$ are such that $\text{Int}(\gamma) - \{\alpha_1, \dots, \alpha_n\} \subset \Omega$; we can choose small circles around $\alpha_1, \dots, \alpha_n$ - say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$; and

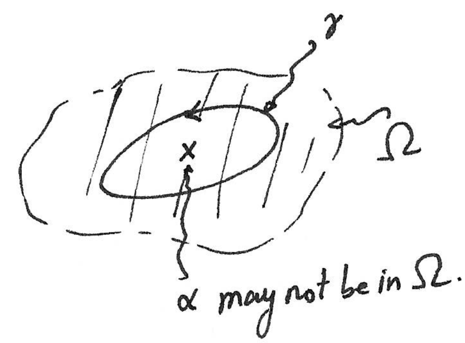
$$\int_{\gamma} f(z)\,dz = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f(z)\,dz$$



$(\text{Int}(\gamma) - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Omega)$

(13.2) A slight generalization of Cauchy's Theorem.

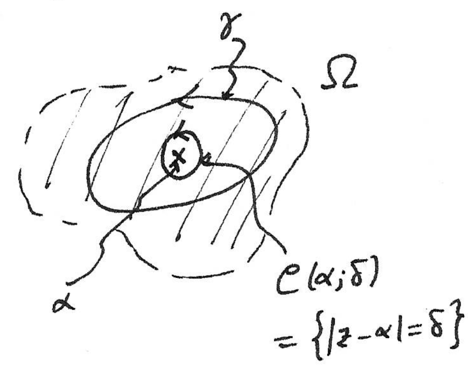
Again, $\Omega \subset \mathbb{C}$ is open, connected. $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable. $\gamma : [a,b] \to \Omega$ is a contour. Assume there is $\alpha \in \text{Interior}(\gamma)$ such that $\text{Interior}(\gamma) - \{\alpha\} \subset \Omega$.



Proposition. – Assume $\lim\limits_{z \to \alpha} (z-\alpha) f(z) = 0$.

Then $\displaystyle\int_\gamma f(z)\,dz = 0$.

Proof. Let $\varepsilon > 0$ be given. Take $r > 0$ be such that $D(\alpha; r) \subset \Omega$ and $|z - \alpha| < r$ implies $|(z-\alpha) f(z)| < \varepsilon$.



Let $0 < \delta < r$. Then $\displaystyle\int_\gamma f(z)\,dz = \int_{\mathcal{C}(\alpha;\delta)} f(z)\,dz$

$\mathcal{C}(\alpha;\delta) =$ (counterclockwise) circle of radius $\delta > 0$ centered at α

by contour deformation principle (13.1)

$$\left| \int_{\mathcal{C}(\alpha;\delta)} f(z)\,dz \right| < \frac{\varepsilon}{\delta} \cdot 2\pi\delta = 2\pi\varepsilon$$

(for $z \in \mathcal{C}(\alpha;\delta)$, $|z-\alpha| = \delta < r$ implies $|f(z)| < \dfrac{\varepsilon}{|z-\alpha|} = \dfrac{\varepsilon}{\delta}$.
$\text{length}(\mathcal{C}(\alpha;\delta)) = 2\pi\delta$ – use inequality from (12.1)).

Hence $\displaystyle\left| \int_\gamma f(z)\,dz \right| < 2\pi\varepsilon$

for every $\varepsilon > 0$, implying that $\displaystyle\int_\gamma f(z)\,dz = 0$. $\square$

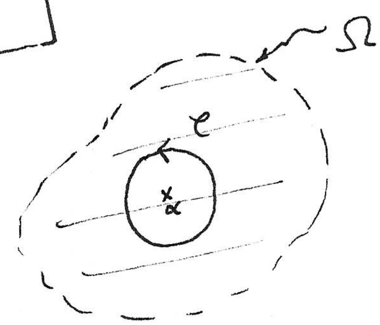
(13.3) Cauchy's integral formula. Let $f : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable. Then f is ($\mathbb{C}$-) differentiable to all orders; and $\forall\, \alpha \in \Omega$ we have

$$\boxed{\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz} \qquad n = 0, 1, 2, \ldots$$

(here $\mathcal{C}$ is a (small) circle of radius r; centered at α. $r > 0$ is such that $\overline{D}(\alpha; r) \subset \Omega$.). $\mathcal{C}$ is counterclockwise oriented. [By contour deformation, $\mathcal{C}$ could be any contour around α; $\mathrm{Int}(\mathcal{C}) \subset \Omega$]

Proof. $n=0$ case:

$$\boxed{f(\alpha) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z-\alpha}\, dz}$$



Proof of $n=0$ case:

Let $g(z) = \dfrac{f(z) - f(\alpha)}{z - \alpha}$.

Then $\lim\limits_{z \to \alpha} (z-\alpha)\, g(z) = 0$ and by Prop. 13.2 we get

$$\int_{\mathcal{C}} g(z)\, dz = 0 \quad \Rightarrow \quad \int_{\mathcal{C}} \frac{f(z)}{z-\alpha}\, dz = f(\alpha) \int_{\mathcal{C}} \frac{dz}{z-\alpha}$$

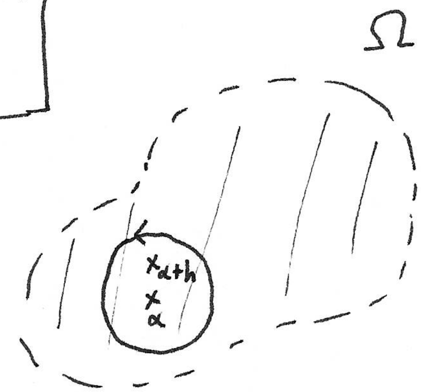
$$\int_{\mathcal{C}} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{\cancel{r e^{it}} \cdot i \cdot dt}{\cancel{r \cdot e^{it}}} = 2\pi i. \quad \text{Hence} \quad \int_{\mathcal{C}} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha).$$

$\mathcal{C} = \alpha + r e^{it}$
$(0 \leq t \leq 2\pi)$

The remainder of the proof is by induction on n. While it is not necessary, I will write $n=1$ case separately, which makes the induction step clear.

Proof of $n=1$ case.

$$\boxed{f'(\alpha) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-\alpha)^2}\,dz}$$



Let $h \in \mathbb{C}$ be such that $|h| < r$

(recall: $\mathcal{C} = \alpha + re^{it}$ $(0 \le t \le 2\pi)$)

Then, by $n=0$ case

$$\frac{f(\alpha+h) - f(\alpha)}{h} = \frac{1}{2\pi i h}\int_{\mathcal{C}} f(z)\left(\frac{1}{z-\alpha-h} - \frac{1}{z-\alpha}\right) dz$$

$$= \frac{1}{2\pi i \cancel{h}}\int_{\mathcal{C}} f(z)\cdot\frac{\cancel{h}}{(z-\alpha-h)(z-\alpha)}\,dz$$

Claim: $$\lim_{h\to 0}\int_{\mathcal{C}} f(z)\frac{dz}{(z-\alpha-h)(z-\alpha)} = \int_{\mathcal{C}}\frac{f(z)}{(z-\alpha)^2}\,dz$$

Proof of the claim: $$\int_{\mathcal{C}} f(z)\left(\frac{1}{(z-\alpha-h)(z-\alpha)} - \frac{1}{(z-\alpha)^2}\right) dz$$

$$= \int_{\mathcal{C}} f(z)\cdot\frac{h}{(z-\alpha-h)(z-\alpha)^2}\,dz.$$ By triangle inequality, modulus of this integral can be bounded as follows. Let $M = \operatorname{Max}\{|f(z)| : z \in \mathcal{C}\}$

For $z \in \mathcal{C}$; $|z-\alpha| = r$. $|z-\alpha-h| \geq |z-\alpha| - |h| = r - |h|$

$$\Rightarrow \quad \left| \frac{f(z)}{(z-\alpha-h)(z-\alpha)^2} \right| \leq \frac{M}{(r-|h|)\, r^2} . \qquad \text{Length}(\mathcal{C}) = 2\pi r$$

Hence
$$\left| \int_{\mathcal{C}} \frac{f(z)}{(z-\alpha-h)(z-\alpha)} dz - \int_{\mathcal{C}} \frac{f(z)}{(z-\alpha)^2} dz \right| \leq \frac{M \cdot |h|}{(r-|h|)\, r^2} \cdot 2\pi r \to 0$$
as $|h| \to 0$.

(end of proof for $n=1$) □

Proof of the induction step. Assume that we have shown

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-\alpha)^{n+1}} dz$$
(γ any contour in Ω; $\alpha \in \text{Int}(\gamma) \subset \Omega$).

The proof of $f^{(n+1)}(\alpha) = \dfrac{(n+1)!}{2\pi i} \displaystyle\int_{\mathcal{C}} \frac{f(z)}{(z-\alpha)^{n+2}} dz$ becomes equivalent to checking the following:

Claim.
$$\lim_{h \to 0} \int_{\mathcal{C}} \left[\frac{f(z)}{h} \cdot \left(\frac{1}{(z-\alpha-h)^{n+1}} - \frac{1}{(z-\alpha)^{n+1}} \right) - \frac{f(z) \cdot (n+1)}{(z-\alpha)^{n+2}} \right] dz = 0.$$

Using binomial formula the integrand simplifies to (I am leaving the details of this easy calculation)

$$\frac{f(z) \cdot h}{(z-\alpha-h)^{n+1} (z-\alpha)^{n+2}} \cdot \sum_{\ell=1}^{n+1} (-1)^{\ell-1} h^{\ell-1} (z-\alpha)^{n+1-\ell} \left((n+1)\binom{n+1}{\ell} - \binom{n+1}{\ell+1} \right)$$

Convention $\binom{n+1}{n+2} = 0$

Hence, we get similar bound and finish the proof

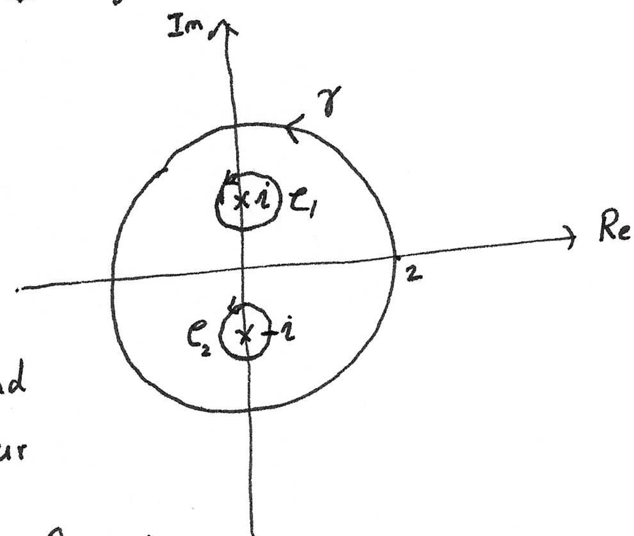
$$\left|\int_{\mathcal{C}}\left[\frac{f(z)}{h}\left(\frac{1}{(z-\alpha-h)^{n+1}}-\frac{1}{(z-\alpha)^{n+1}}\right)-\frac{(n+1)\,f(z)}{(z-\alpha)^{n+2}}\right]dz\right|$$

$$\leqslant \frac{M\cdot|h|}{(r-|h|)^{n+1}\,r^{n+2}}\cdot 2\pi r\cdot\left\{\sum_{l=1}^{n+1}|h|^{l-1}\,r^{n+1-l}\left((n+1)\binom{n+1}{l}-\binom{n+1}{l+1}\right)\right\}$$

$\longrightarrow 0$ as $|h|\to 0$ □

(13.4) From now on, $\mathbb{C}$-differentiable functions will be called <u>holomorphic</u>.
(to remember "once differentiable, always differentiable" property).

Example. $f(z)=\dfrac{e^z}{z^2+1}$; $\Omega=\mathbb{C}-\{\pm i\}$. Let γ be counterclockwise circle of radius 2 centered at 0.



Compute $\int_\gamma f(z)\,dz$.

<u>Sol</u>: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles around i and $-i$ (of radius <1). Then by contour deformation

$$\int_\gamma f(z)\,dz=\int_{\mathcal{C}_1} f(z)\,dz+\int_{\mathcal{C}_2} f(z)\,dz$$

$$\int_{\mathcal{C}_1}\frac{e^z}{z^2+1}\,dz=\int_{C_1}\left(\frac{e^z}{z+i}\right)\frac{dz}{z-i}=2\pi i\left(\frac{e^i}{i+i}\right)=\pi\cdot e^i.$$

holomorphic within C_1

Cauchy's integral formula for $n=0$.

$$\int_{C_2} \frac{e^z}{z^2+1}\,dz = \int_{C_2} \underbrace{\left(\frac{e^z}{z-i}\right)}_{\text{holomorphic within } C_2} \frac{dz}{z+i} \overset{\text{Cauchy's integral formula (for } n=0)}{=} 2\pi i\,\frac{e^{-i}}{-i-i} = -\pi e^{-i}.$$

Hence
$$\int_\gamma \frac{e^z}{z^2+1}\,dz = \pi\left(e^{i}-e^{-i}\right)$$
$$= 2\pi i\,\frac{e^{i}-e^{-i}}{2i} = 2\pi i \sin(1). \qquad \square$$

(13.5) A corollary of Cauchy's Theorem (12.5) and Morera's Thm. (11.6).

Let $\Omega \subset \mathbb{C}$ be an open, connected and simply-connected set.

(i) Then every holomorphic function $f : \Omega \to \mathbb{C}$ admits an antiderivative (since $\int_\gamma f(z)\,dz = 0$ for every contour γ (by Cauchy's Thm), f admits an antiderivative - by Morera's Theorem)

(ii) Let $u \in C^2(\Omega)$ be a <u>harmonic function</u>. (recall - this means $\exists\, u:\Omega\to\mathbb{R}$, $u_x, u_y, u_{xx}, u_{xy}, u_{yx}, u_{yy}$ all exist & are continuous; and $u_{xx}+u_{yy}=0$)

Then $\exists\, v \in C^2(\Omega)$ such that $u+iv : \Omega \to \mathbb{C}$ is holomorphic.

In particular $u \in C^\infty(\Omega)$ is infinitely differentiable.

(Hint - Let $g = u_x - i u_y$. Verify (using Thm. 9.1) that g is holomorphic. Use (i) to get f s.t. $f' = g$. Take $v = \mathrm{Im}(f)$.)