(14.0) Recall that we have proved the following important properties of line integrals and holomorphic functions.

Let $\Omega \subset \mathbb{C}$ be an open, connected subset and $f : \Omega \to \mathbb{C}$ a continuous fn.

(i) $f = u + iv$ is holomorphic $\iff$ $u, v \in C^1(\Omega)$ and $u_x = v_y$, $u_y = -v_x$

[Thm 9.1 + Thm. 13.3]

In this case $f' = u_x + i v_x = u_x - i u_y = v_y - i u_y = v_y + i v_x$.

(ii) Let $\gamma : [a,b] \to \Omega$ be a piecewise smooth path. Then

[Triangle ineq. (12.1)] $\left| \int_\gamma f(z)\,dz \right| \leq M \cdot L$ where $M = \mathrm{Max}\{|f(\gamma(t))| : a \leq t \leq b\}$, $L = \mathrm{Length}(\gamma)$

(iii) The following are equivalent:

[Morera's Thm (11.6)]

- f admits an antiderivative
- $\int_\gamma f(z)\,dz = 0$ for every closed path γ in Ω.
- $\int_\gamma f(z)\,dz = 0$ for every rectangular path in Ω (i.e. boundary of a (closed) rectangle).

(iv) $\int_\gamma f(z)\,dz = 0$ in the following two cases (assuming f is holomorphic):

[Cauchy's Thm ~~(13.3)~~ (12.5)]

- Ω is simply connected; γ is any closed path
- γ is simple, closed and Interior$(\gamma) \subset \Omega$.

<u>Defn.</u> A <u>contour</u> is a simple, closed (piecewise smooth) path. By default, we will assume that a contour is counterclockwise oriented.

(v) $$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$ where
- γ_1 is a contour in Ω
- γ_2 is a contour in Interior(γ_1)
- Interior$(\gamma_1) \cap$ Exterior$(\gamma_2) \subset \Omega$
- f is holomorphic

[Principle of contour deformations (13.1)]

(vi) f is holomorphic; γ a contour in Ω; $\alpha \in$ Interior$(\gamma) \subset \Omega$.

Then
$$\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

[Cauchy's integral formula (13.3)]

(14.1) Some easy consequences of these results.

(1) If $f : \Omega \to \mathbb{C}$ is holomorphic and Ω is simply-connected then f admits an antiderivative $F : \Omega \to \mathbb{C}$ given by fixing a point $\alpha \in \Omega$ and defining $F(z_0) = \int_\gamma f(z)\,dz$ (γ : any path joining α to z_0).

(2) Let Ω be a simply-connected domain (open and connected as always). Let $u : \Omega \to \mathbb{R}$ be a harmonic function (i.e. $u \in C^2(\Omega)$ and $u_{xx} + u_{yy} = 0$). Then $\exists\ v : \Omega \to \mathbb{R}$ s.t. $u + iv : \Omega \to \mathbb{C}$ is holomorphic. In particular, u is infinitely differentiable.

(Proof. Define $g(x+yi) := u_x(x,y) - i\,u_y(x,y) : \Omega \to \mathbb{C}$. Then $\mathrm{Re}(g), \mathrm{Im}(g) \in C^1(\Omega)$ and Cauchy-Riemann equations hold for g. $\Rightarrow g$ is holomorphic. Since Ω is ~~con~~ simply-connected, g admits an antiderivative f. Let us write $f(z) = P(x,y) + i\,Q(x,y)$.

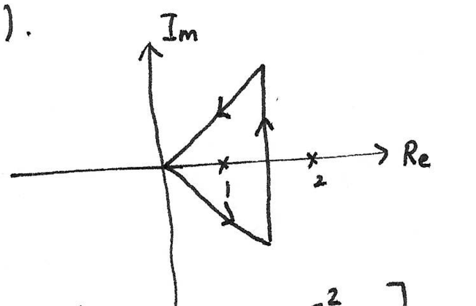
Then we get $P_x = u_x$ and $P_y = u_y$ $\Rightarrow P(x,y) = u(x,y) + \underset{\uparrow\ \text{constant}}{C}$

(from $f' = g = P_x - i P_y$)

Replace P by $P - C$ to ensure that $\mathrm{Re}(f) = u$. $\square$)

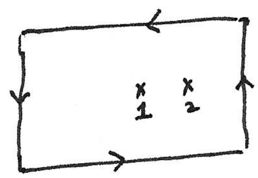
(14.2) Examples. I. $\displaystyle\int_C \frac{\sin(z)}{z-1}\,dz$ where C is a contour containing 1 in its interior

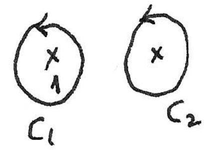
$$= 2\pi i \sin(1).$$

II. $\displaystyle\int_C \frac{e^{z^2}}{(z-1)^2(z-2)}\,dz$ $\Big($ C =  $\Big)$.

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{e^{z^2}}{z-2}\right)\right]_{\text{set } z=1} = 2\pi i \left[\frac{e^{z^2}\cdot 2z}{z-2} - \frac{e^{z^2}}{(z-2)^2}\right]_{\text{set } z=1}$$

$$= 2\pi i\, e\,(-2-1) = -6\pi i e.$$

III. $\displaystyle\int_C \frac{z}{(z-1)(z-2)^2}\,dz$ where C = 

$$= \int_{C_1} \frac{z}{(z-2)^2}\frac{dz}{z-1} + \int_{C_2} \frac{z}{z-1}\frac{dz}{(z-2)^2}$$ where 

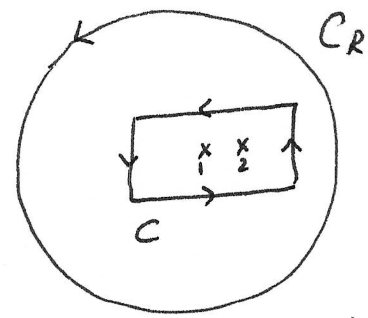
$$\int_{C_1} \frac{z}{(z-2)^2}\frac{dz}{z-1} = 2\pi i\left[\frac{z}{(z-2)^2}\right]_{\text{set } z=1} = 2\pi i$$

$$\int_{C_2} \frac{z}{z-1}\frac{dz}{(z-2)^2} = 2\pi i\left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]_{z=2} = 2\pi i\left[\frac{-1}{(z-1)^2}\right]_{z=2} = -2\pi i$$

$$\Rightarrow \int_C \frac{z}{(z-1)(z-2)^2}\,dz = 0.$$

IV. A clever way of working out example III above.

Let $R \in \mathbb{R}_{>0}$ be large enough so that C_R = circle (counterclockwise) of radius R centered at 0 contains C in its interior.



Then $$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = \int_{C_R} \frac{z}{(z-1)(z-2)^2}\,dz$$

independent of R

For z on C_R (i.e. $|z| = R$), $$\left|\frac{z}{(z-1)(z-2)^2}\right| \leq \frac{R}{(R-1)(R-2)^2}$$ (by triangle ineq. $|z-1| \geq |z|-1$, $|z-2| \geq |z|-2$)

$$\Rightarrow \left|\int_C \frac{z}{(z-1)(z-2)^2}\,dz\right| \leq \frac{R}{(R-1)(R-2)^2} \cdot 2\pi R \to 0 \text{ as } R \to \infty.$$

Hence $$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = 0.$$

Remark.- This method will work for any rational function when degree (numerator) $\leq$ degree (denominator) -2.

(14.3) Liouville's Theorem*.- Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that f is bounded (i.e. $\exists\, M \in \mathbb{R}_{>0}$ s.t. $|f(z)| \leq M$ for every $z \in \mathbb{C}$). Then f is a constant.

---

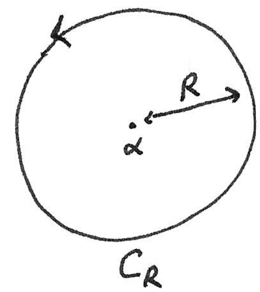
* Joseph Liouville (24/3/1809 – 8/9/1882). This theorem is actually due to Cauchy (1844). It was called Liouville's Thm. by Borchardt who learnt about it from Liouville's lectures in 1847.

Proof. We will show that $f'(\alpha) = 0$ for every $\alpha \in \mathbb{C}$. This implies that f is a constant (see Lemma 9.5).

So, given $\alpha \in \mathbb{C}$, let $C_R = C(\alpha; R) =$ counterclockwise circle of radius R centered at α ($R \in \mathbb{R}_{>0}$)

By Cauchy's integral formula

$$f'(\alpha) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-\alpha)^2}\, dz$$



By triangle inequality $\left| \int_{C_R} \frac{f(z)}{(z-\alpha)^2}\, dz \right| \leq \frac{M}{R^2} \cdot 2\pi R$

$\Rightarrow |f'(\alpha)| \leq \frac{M}{R} \to 0$ as $R \to \infty$. Hence $|f'(\alpha)| = 0 \Rightarrow f'(\alpha) = 0$ □

(14.4) Liouville's Theorem (as it is called now) was obtained by Cauchy in 1844 in order to give a new proof of the fundamental theorem of algebra.

Fundamental Theorem of Algebra. - Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients $(a_0, \dots, a_n \in \mathbb{C};\ a_n \neq 0)$. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof. (for simplicity assume $a_n = 1$.) The proof is by contradiction.
i.e. replace p by $\frac{p}{a_n}$

Assume that $p(z_0) \neq 0$ for every $z_0 \in \mathbb{C}$. Then $f(z) = \dfrac{1}{p(z)}$ is a holomorphic function defined on the entire complex plane: $f : \mathbb{C} \to \mathbb{C}$.

<u>Claim</u>: f is bounded. (Given this claim, f and hence p will be constant functions, contradicting the assumption that degree$(p) = n \geq 1$).

<u>Proof of the claim:</u> $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$ implies, by triangle inequality that
$$|p(z)| \geq |z|^n - \sum_{j=0}^{n-1} |a_j| \cdot |z|^j$$
$$= |z|^n \left(1 - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|^{n-j}} \right)$$

Choose $R > 0$ such that $\dfrac{|a_j|}{R^{n-j}} \leq \dfrac{1}{2n}$ (for instance). Then for $|z| > R$, we have $|p(z)| > R^n \left(1 - \sum_{j=0}^{n-1} \frac{1}{2n} \right) = \frac{R^n}{2}$.

$\Rightarrow$ for $|z| > R$, $|f(z)| = \dfrac{1}{|p(z)|} < \dfrac{2}{R^n}$.

Now let $M = \operatorname{Max} \{ |f(z)| : z \in \overline{D}(0; R) \}$ (recall $\overline{D}(0;R) = \{ |z| \leq R \}$)

Then $|f(z)| \leq \operatorname{Max} \left\{ M, \frac{2}{R^n} \right\} \Rightarrow f$ is bounded.
$(\forall z \in \mathbb{C})$

$\square$

(14.5) Corollary. - Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ be a polynomial with complex coefficients. Then $\exists\ \alpha_1, \dots, \alpha_n \in \mathbb{C}$ (not necessarily distinct) so that

$$p(z) = (z-\alpha_1) \cdots (z-\alpha_n).$$

Proof. Given $\alpha \in \mathbb{C}$, we can divide $p(z)$ by $(z-\alpha)$ to write

$$p(z) = (z-\alpha)\,q(z) + r \ ; \ \deg(q) = n-1 \ ; \ r \in \mathbb{C}.$$

This proves that $p(\alpha)=0 \iff z-\alpha$ divides $p(z)$ (i.e. $r=0$). Let α_1 be a root of $p(z)=0$. The corollary follows from induction argument - applied to $q(z) = \dfrac{p(z)}{z-\alpha_1}$. $\square$

(14.6) Some historical remarks (optional). - The fundamental theorem of algebra was first conjectured in 1629 by Albert Girard. The first serious attempt towards its proof was made by d'Alembert in 1746. He gave two arguments - the first one based on "formal solutions" which was entirely flawed - in that it eventually ends up assuming FTA; while the second one was almost precise an goes as follows:

- For $R>0$ sufficiently large $|p(z)|$ grows as R^n $(p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0)$ hence cannot be zero.
- On $\overline{D}(0,R) = \{|z| \leq R\}$, $|p(z)|$ must attain its minimum somewhere.
- if $|p(z_0)| = \text{Min}\{|p(z)| : z \in \overline{D}(0;R)\}$ is not zero, we can make it smaller by changing z_0 to $z_0+\delta$ for small enough δ.

We will see a precise proof of this next time

(This "proof" of d'Alembert was elaborated by Argand in 1814).

- In 1749, Euler sketched an argument to show that every real polynomial can be written as a product of linear and quadratic factors - based on induction on k where degree of polynomial $= 2^k \cdot \ell$ (ℓ is odd). This reasoning was turned into a proof by Lagrange in 1772 and is based on existence of "splitting extensions" -

- (Splitting extensions) Given a polynomial with entries from a field; there exists a larger field (extension) where the given polynomial splits into a product of linear factors.

The statement written above was later proved by Kronecker and Frobenius in 1870's - but until then the proof of Euler-Lagrange was in doubt.

- In 1799, Gauss gave another proof of FTA (his dissertation). Gauss' original proof goes as follows. $(p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0)$

(i) Note that $\quad Re(z^n) = 0 \iff \arg(z) = \dfrac{(2k-1)\pi}{2n} \pmod{2\pi}$; $k = 1, 2, \ldots, 2n$. —(1)

$Im(z^n) = 0 \iff \arg(z) = \dfrac{2k\pi}{2n}$; $k = 0, 1, \ldots, 2n-1$ —(2)

(ii) As $\lim\limits_{|z|\to\infty} \left|\dfrac{p(z)}{z^n}\right| = 1$, for z lying on the curves $Re(p(z)) = 0$; $\arg(z)$ ($Im(p(z)) = 0$) gets closer to the ones listed in (1) ((2)). Hence, for $R > 0$ sufficiently large, the curves $Re(p(z)) = 0$ & $Im(p(z)) = 0$ meet the circle $|z| = R$ alternately.

(iii) Within the circle, the solid lines must join with solid lines and dotted lines with dotted. Hence they must cross each other at some z_0 where

$$Re(p(z_0)) = Im(p(z_0)) = 0$$

i.e. $p(z_0) = 0$

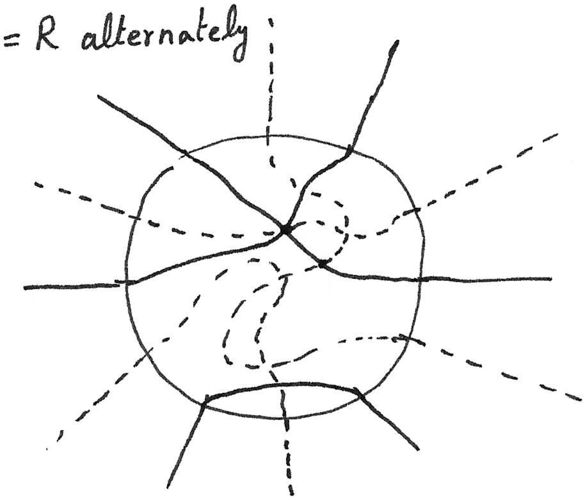


Illustration for $n=3$

—— $Im(p(z)) = 0$

- - - - $Re(p(z)) = 0$