Lecture 15

Recall - we are discussing some applications of Cauchy's theorems.

Last time we proved:

<u>Liouville's Theorem</u>: Bounded holomorphic functions defined on the entire complex plane are constants.

<u>Fundamental Theorem of Algebra</u> Every degree n polynomial over $\mathbb{C}$ is a product of linear factors.

(15.1) Rational functions and their integrals.

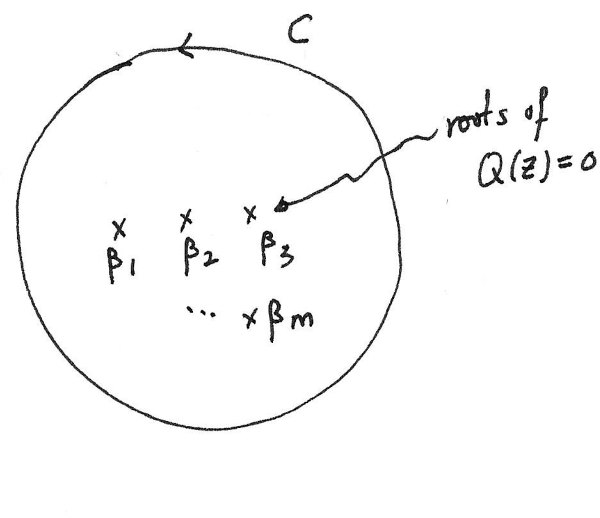
Let us consider two polynomials $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ $(a_n \neq 0)$ and $Q(z) = z^m + b_{m-1} z^{m-1} + \dots + b_m$. (I am assuming the leading coefficient in Q is 1).

Assume that $n < m$. We have the following analogue of the example (14.2) - IV (Lecture 14, page 4).

Lemma. Let C be a contour such that all the roots of $Q(z)=0$ are in the interior of C.

Then

$$\frac{1}{2\pi i} \int_C \frac{P(z)}{Q(z)} = \begin{cases} 0 & \text{if } n \leq m-2, \\ a_n & \text{if } n = m-1. \end{cases}$$



I will leave the proof using estimates and triangle inequality – akin to the one from (14.2) – IV – as an easy exercise.

Here I will write a different argument using substitution $w = z^{-1}$.

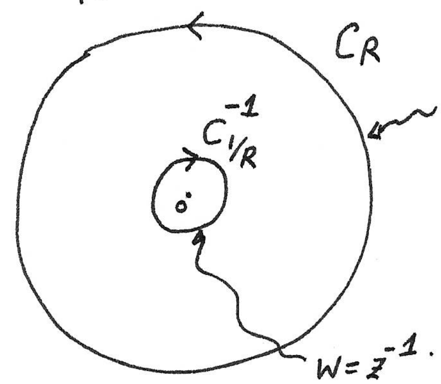
Proof. Set $w = z^{-1}$ so that $dw = -z^{-2}dz$, i.e. $dz = -w^{-2}dw$.

Assuming $C = C_R = \{Re^{i\theta} : 0 \le \theta \le 2\pi\}$ where

$R > \mathrm{Max}\{|\beta_j| : 1 \le j \le m\}$ $\quad (Q(z) = (z-\beta_1)\cdots(z-\beta_m))$

$z \in C_R \iff w = \frac{1}{R}e^{-i\theta}$ $(0 \le \theta \le 2\pi)$ traverses clockwise circle of radius $\frac{1}{R}$ around 0

$$\int_{C_R} \frac{P(z)}{Q(z)}\,dz = \int_{C^{-1}_{\frac{1}{R}}} \frac{P(\frac{1}{w})}{Q(\frac{1}{w})}(-w^{-2})\,dw$$



$$= \int_{C_{1/R}} \frac{\frac{a_n}{w^n} + \frac{a_{n-1}}{w^{n-1}} + \dots + \frac{a_1}{w} + a_0}{\frac{1}{w^m} + \frac{b_{m-1}}{w^{m-1}} + \dots + \frac{b_1}{w} + b_0}\,\frac{dw}{w^2}$$

$$= \int_{C_{1/R}} w^{m-n-2}\,\frac{a_n + a_{n-1}w + \dots + a_0 w^n}{1 + b_{m-1}w + \dots + b_0 w^m}\,dw = 0 \text{ if } m-n-2 \ge 0$$

$$= 2\pi i \left[\frac{a_n + a_{n-1}w + \dots + a_0 w^n}{1 + b_{m-1}w + \dots + b_0 w^m}\right]_{w=0} = 2\pi i\, a_n \text{ if } m-n-2 = -1$$

□

(15.2) Some comments on partial fractions.

Again we consider a rational function $f(z) = \dfrac{P(z)}{Q(z)}$ where

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 \quad (a_n \neq 0)$$
$$Q(z) = z^m + b_{m-1} z^{m-1} + \dots + b_0$$
; assume $n < m$.

- Special case : assume $Q(z) = (z-\beta_1) \cdots (z-\beta_m)$ where $\beta_1, \dots, \beta_m$ are <u>distinct</u>. In this case - the partial fraction decomposition of $f(z) = \dfrac{P(z)}{Q(z)}$ takes the following form.

$$(*) \quad \frac{P(z)}{(z-\beta_1)\cdots(z-\beta_m)} = \sum_{j=1}^{m} \frac{A_j}{z-\beta_j} \quad ; \text{ where } A_1, \dots, A_m \in \mathbb{C}.$$

The numbers $A_1, \dots, A_m$ can be determined by

- either multiply both sides of $(*)$ by $z-\beta_k$ and set $z = \beta_k$;
- or $\displaystyle\oint_{\beta_k} \frac{P(z)}{Q(z)}\,dz = 2\pi i\, A_k$.

In either case, we get $\quad A_k = \dfrac{P(\beta_k)}{\displaystyle\prod_{\substack{l=1,\dots,m \\ l \neq k}} (\beta_k - \beta_l)} \quad (\forall\ 1 \leq k \leq m).$

• General case. - $Q(z) = (z-\beta_1)^{m_1} \cdots (z-\beta_r)^{m_r}$

$\beta_1, \ldots, \beta_r \in \mathbb{C}$ are distinct ; $m_1, \ldots, m_r \in \mathbb{Z}_{\geq 1}$

and $m_1 + \ldots + m_r = m$.

The partial fraction decomposition of $f(z) = \dfrac{P(z)}{Q(z)}$ in this case takes the following form:

$$\frac{P(z)}{Q(z)} = \left(\frac{A_1^{(1)}}{z-\beta_1} + \frac{A_1^{(2)}}{(z-\beta_1)^2} + \ldots + \frac{A_1^{(m_1)}}{(z-\beta_1)^{m_1}} \right) + \left(\frac{A_2^{(1)}}{z-\beta_2} + \ldots + \frac{A_2^{(m_2)}}{(z-\beta_2)^{m_2}} \right)$$

$$+ \ldots + \left(\frac{A_r^{(1)}}{z-\beta_r} + \ldots + \frac{A_r^{(m_r)}}{(z-\beta_r)^{m_r}} \right)$$

$$= \sum_{k=1}^{r} \left(\sum_{l=1}^{m_k} \frac{A_k^{(l)}}{(z-\beta_k)^l} \right)$$

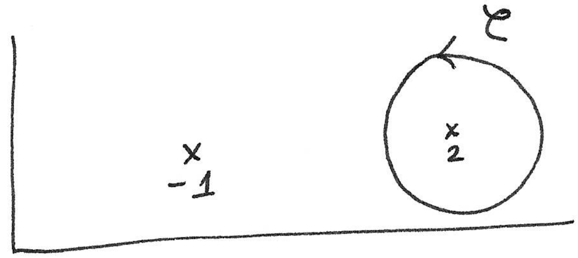
By Cauchy's integral formula ; $A_k^{(l)}$ can be computed as follows:

$$A_k^{(l)} = \frac{1}{2\pi i} \int_{C_k} (z-\beta_k)^{l-1} \frac{P(z)}{Q(z)} \, dz$$

where C_k is a small contour around β_k (small so that β_l is outside C_k for $l \neq k$).

(15.3) Example. Let $f(z) = \dfrac{z^2+2}{(z+1)(z-2)^2}$.

Then $f(z) = \dfrac{A}{z+1} + \dfrac{B}{z-2} + \dfrac{C}{(z-2)^2}$. Let us determine the value of B. Let $\mathcal{C}$ be a counterclockwise circle around 2 of radius < 3. Then



$$\frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\,dz = \frac{1}{2\pi i}\left\{ \underbrace{\cancel{\int_{\mathcal{C}} \frac{A}{z+1}\,dz}}_{0} + \int_{\mathcal{C}} \frac{B}{z-2}\,dz + \underbrace{\cancel{\int_{\mathcal{C}} \frac{C}{(z-2)^2}\,dz}}_{0} \right\}$$

$$= B.$$

$$\Rightarrow \quad B = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{z^2+2}{(z+1)(z-2)^2}\,dz = \left[\frac{d}{dz}\left(\frac{z^2+2}{z+1}\right)\right]_{z=2}$$

$$= \left[\frac{2z}{z+1} - \frac{z^2+2}{(z+1)^2}\right]_{z=2} = \frac{4}{3} - \frac{6}{9} = \frac{4}{3} - \frac{2}{3}$$

$$= \frac{2}{3}.$$

The values of A and C can be computed by "clearing denominators"

$$A = \left[(z+1)\,f(z)\right]_{z=-1} = \left[\frac{z^2+2}{(z-2)^2}\right]_{z=-1} = \frac{3}{9} = \frac{1}{3}.$$

$$C = \left[(z-2)^2 f(z)\right]_{z=2} = \left[\frac{z^2+2}{z+1}\right]_{z=2} = \frac{6}{3} = 2.$$