Lecture 16

(16.0) Recall that we have been exploring applications of Cauchy's integral formula:

$$\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

$f : \Omega \to \mathbb{C}$ is holomorphic; $\Omega \subset \mathbb{C}$ is an open, connected set; $\gamma : [a,b] \to \Omega$ is a contour (counterclockwise) such that $\alpha \in \text{Interior}(\gamma) \subset \Omega$.

(16.1) Mean Value Property.

Let $R \in \mathbb{R}_{>0}$ be such that the closed disc $\overline{D}(\alpha;R) \subset \Omega$.

Take $\gamma = C(\alpha;r)$ $\left(\gamma(\theta) = \alpha + Re^{i\theta} \; (0 \leq \theta \leq 2\pi) \right)$.

Then $n=0$ case of Cauchy's integral formula becomes

$$f(\alpha) = \frac{1}{2\pi i} \int_{C(\alpha;R)} \frac{f(z)}{z-\alpha}\, dz$$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(\alpha + Re^{i\theta})}{\cancel{R}\cancel{e^{i\theta}}} \cancel{R} \cdot \cancel{e^{i\theta}} \cdot \cancel{i} \cdot d\theta$$

i.e. $$\boxed{f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + Re^{i\theta})\, d\theta}$$

Mean Value Property of holomorphic functions

Remark. - This equation $f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + Re^{i\theta})\, d\theta$

is interpreted as "value of a holomorphic function at a point is the average of its values on a sorrounding circle".

Taking real parts (or imaginary) parts of this identity gives
$(u = \mathrm{Re}(f))$

$$u(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(\alpha + Re^{i\theta})\, d\theta \qquad (u : \Omega \to \mathbb{R})$$

or, in (x,y) terms - if $\alpha = a + bi$;

$$\boxed{u(a,b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + R\cos(\theta), b + R\sin(\theta))\, d\theta}$$

Therefore we obtain the following

<u>Theorem (Mean Value Property)</u>. Every <u>holomorphic</u> $f : \Omega \to \mathbb{C}$, and every <u>harmonic</u> $u : \Omega \to \mathbb{R}$ has mean value property.

(for harmonic functions, the equation $u(a,b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + R\cos(\theta), b + R\sin(\theta))\, d\theta$ is due to Gauss - it is called Gauss' mean value theorem. The holomorphic case is (again) due to Cauchy).

Later we will show that MVP (abbreviation for mean value property) characterizes harmonic functions. Meaning, if $u : \Omega \to \mathbb{R}$ is continuous s.t. $u(a,b) = \frac{1}{2\pi}\int_0^{2\pi} u(a+R\cos(\theta), b+R\sin(\theta))\, d\theta$

$(\forall\ (a,b) \in \Omega$ and $R \in \mathbb{R}_{>0}$ such that $\overline{D}(\alpha; R) \subset \Omega)$, then u is harmonic.

Now we will prove that MVP implies maximum modulus principle.

(16.2) Theorem. Let $\Omega \subset \mathbb{C}$ be open and connected. Let $g : \Omega \to \mathbb{C}$ be any continuous function that has MVP. Assume that there is $\alpha \in \Omega$ such that $|g|$ has local maximum at α (meaning, $\exists\ r > 0$ s.t. $|g(\alpha)| \geq |g(z)|$ for every $z \in D(\alpha; r)$). Then g is constant near α.

(Meaning, $\exists\ \rho > 0$ s.t. $g(\alpha) = g(z)$ for every $z \in D(\alpha; \rho)$).

Proof.- If $g(\alpha) = 0$, there is nothing to prove since in this case, $\forall\ z \in D(\alpha; r)$, $|g(z)| \leq |g(\alpha)| = 0 \Rightarrow g(z) = 0 = g(\alpha)$.

Let us assume $g(\alpha) \neq 0$. Replacing g by $e^{-i\arg(g(\alpha))} \cdot g$ we may also assume that $g(\alpha) \in \mathbb{R}_{>0}$.

Let $r \in \mathbb{R}_{>0}$ be as in the statement of the theorem (i.e. $|g(z)| \leq g(\alpha)$ for every $z \in D(\alpha; r)$).

For each $0 < \rho < r$, let

$$M(\rho) = \mathrm{Max}\left\{ |g(\alpha + \rho \cdot e^{i\theta})| : 0 \leq \theta \leq 2\pi \right\}$$

Then, by MVP,

$$g(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha + \rho e^{i\theta})\, d\theta.$$

Combined with our assumption of local max. at α, we get ($M(\rho) \leq g(\alpha)$):

$$M(\rho) \leq g(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha + \rho e^{i\theta})\, d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |g(\alpha + \rho e^{i\theta})|\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M(\rho)\, d\theta = M(\rho).$$

Hence $g(\alpha) = M(\rho)$ for every $0 < \rho < r$. Now consider

$h(z) := \mathrm{Re}\,(g(\alpha) - g(z)) = g(\alpha) - \mathrm{Re}\,(g(z))$ (since we are assuming $g(\alpha) \in \mathbb{R}_{>0}$)

as a continuous function on $C(\alpha; \rho)$ = circle of radius ρ centered at α.

$h : C(\alpha, \rho) \to \mathbb{R}$. Note (by $|g(z)| \leq g(\alpha)$), $h(z) \geq 0$ and

$$h(z) = 0 \iff g(z) = g(\alpha).$$

Since $\int_0^{2\pi} h(\alpha + \rho \cdot e^{i\theta})\, d\theta = 0$ (by MVP of g), we get that $h(\alpha + \rho \cdot e^{i\theta}) = 0$ ($\forall\ 0 \le \theta \le 2\pi$) ; i.e. $g(z) = g(\alpha)$ for every z on $C(\alpha; \rho)$. This holds for every $\rho \in [0, r)$, hence $g(\alpha) = g(z)$ $\forall\ z \in D(\alpha; r)$. $\square$

(16.3) In the statement of Theorem 16.2, g could be real-valued. For real-valued function $g : \Omega \to \mathbb{R}\ (\subset \mathbb{C})$, we can drop $|\cdot|$ -and the same proof shows that near a local max or local min of g, g is constant.

Theorem 16.2, for holomorphic functions has a stronger version which we will prove later. Namely, we will show (using Taylor Series) that if $f_1, f_2 : \Omega \to \mathbb{C}$ are two holomorphic functions, defined on an open and connected set $\Omega \subset \mathbb{C}$, and if there is a convergent sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ such that $f_1(z_n) = f_2(z_n)$ for every $n \ge 1$, then $f_1(z) = f_2(z)$ $\forall\ z \in \Omega$.

Once we prove this, Thm 16.2 will imply that: if a holomorphic $f : \Omega \to \mathbb{C}$ has a local max for $|f|$, then f is constant.

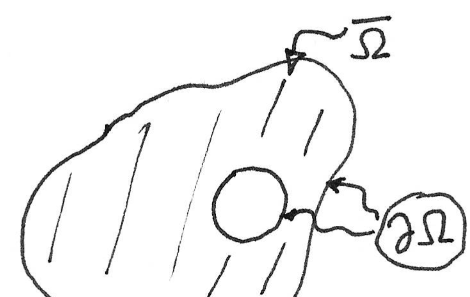
(16.4) Theorem 16.2 has the following corollary, often phrased as saying that "the absolute maximum of $|g|$ occurs at the boundary" – <u>Maximum Modulus Principle</u>. The set up is as follows.

- $\Omega \subset \mathbb{C}$ is open, connected and <u>bounded</u>. Thus its closure $\overline{\Omega}$ is compact. Let $\partial\Omega = \overline{\Omega} \setminus \Omega$ be its boundary.
- $g : \overline{\Omega} \to \mathbb{C}$ is a continuous function whose restriction to Ω has MVP.

Let $M = \mathrm{Max}\{|g(z)| : z \in \partial\Omega\}$ (absolute max. of $|g|$ on the boundary of Ω).

<u>Cor.</u> (i) $|g(z)| \le M$ for every $z \in \Omega$.

(ii) If $|g(z_0)| = M$ for some $z_0 \in \Omega$, then g is constant.



Proof. Let $M' = \mathrm{Max}\{|g(z)| : z \in \overline{\Omega}\} \ge M$.

Since $\overline{\Omega}$ is compact, there exists $\alpha \in \overline{\Omega}$ such that $|g(\alpha)| = M'$.

If $\alpha \in \partial\Omega$, then $M' = M$ and $|g(z)| \le M' = M$ for every $z \in \Omega$, and we are done with (i).

If $\alpha \in \Omega$, then we consider $\mathcal{U} = \{z \in \Omega : g(z) = g(\alpha)\}$ ($\mathcal{U} \ne \phi$ since $\alpha \in \mathcal{U}$)

As absolute max. is also local max, Thm (16.2) implies that $\mathcal{U} \subset \Omega$ is open.

But $\Omega \setminus \mathcal{U} = \{z \in \Omega : g(z) \ne g(\alpha)\} = g^{-1}(\mathbb{C} - \{g(\alpha)\})$ is also open (recall: inverse image, under a continuous function, of an open set is open - Lecture 7, page 5, Prop. (7.5)).

As Ω is connected, we get that $\mathcal{U} = \Omega$, i.e. $g(z) = g(\alpha)$ $\forall z \in \Omega$.
(see Problem 6 of Set 2)

By continuity of g, we get that $g(z) = g(\alpha)$ $\forall z \in \overline{\Omega}$. Hence

$M = M' = |g(\alpha)|$. $\square$

(16.5) With the same set up of Cor (16.4) – i.e., Ω is open, connected and bounded, assume $g : \overline{\Omega} \to \mathbb{R}$ is continuous and has MVP on Ω. In this case (i.e., real-valued functions), Cor. 16.4 takes the following form:

Let
$$m = \operatorname{Min}\{g(z) : z \in \partial\Omega\}$$
$$M = \operatorname{Max}\{g(z) : z \in \partial\Omega\}$$

Then (1) $m \leq g(z) \leq M$ $\forall z \in \Omega$.

(2) If any of the $\leq$, is $=$ for some $z_0 \in \Omega$, then g is constant (and hence $m = M$).