# Lecture 17



(17.0) Recall that last time we proved that every holomorphic (and harmonic) function has <u>mean value property</u>. We used this property to prove the <u>maximum modulus principle</u>. (see Theorem 16.2 and Corollary 16.4).

Another variant of this principle that we can prove at this stage is the following.

<u>Corollary (of Theorem 16.2)</u>. Let $\Omega$ be an open and connected set and $f: \Omega \to \mathbb{C}$ be a holomorphic function. If there exists $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for every $z \in \Omega$, then $f$ is constant on $\Omega$.

(As remarked in Lecture 16, §16.3 - the conclusion of this corollary is valid under weaker hypothesis - and will be proved in that generality later, using Taylor series.)

Proof. (same as the proof of Cor. 16.4 - Lecture 16, page 6).

Let $\mathcal{U} = \{z \in \Omega : f(z) = f(z_0)\} \subset \Omega$. ($\mathcal{U} \neq \phi$ since $z_0 \in \mathcal{U}$).

By Theorem 16.2, $\mathcal{U}$ is an open set.

As inverse image of an open set is open; $\Omega \setminus \mathcal{U} = f^{-1}(\mathbb{C} - \{f(z_0)\})$ is also open.

Hence, $\mathcal{U} = \Omega$ since $\Omega$ is assumed to be connected. $\square$

(17.1) Schwarz' Lemma.* - Let $\mathbb{D} = D(0;1)$ denote the open disc of radius 1 centered at 0.

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that

(i) $f(0) = 0$ ; and (ii) $|f(z)| < 1$ (i.e., $f(\mathbb{D}) \subset \mathbb{D}$).

Then (1) $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$.

(2) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$, then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $f(z) = \lambda z \ \forall z \in \mathbb{D}$.

Proof. Consider the function $g(z) = \begin{cases} \dfrac{f(z)}{z} & ; z \neq 0 \\ f'(0) & ; z = 0 \end{cases}$.

Easy exercise : check that $g : \mathbb{D} \to \mathbb{C}$ is holomorphic.

Now, for $z \in \mathbb{D}$, $|g(z)| = \dfrac{|f(z)|}{|z|} < \dfrac{1}{|z|}$. Meaning, for every $r \in [|z|, 1)$ ; $|g(z)| < \frac{1}{r}$. Letting $r \to 1^-$, we get $|g(z)| \leq 1$. In other words, $|f(z)| \leq |z| \ \forall z \in \mathbb{D}$, proving (1).

If $\exists \ z_0 \in \mathbb{D}$, such that $|f(z_0)| = |z_0|$, we get $|g(z_0)| = 1$, i.e. $|g(z_0)| \geq |g(z)| \ \forall z \in \mathbb{D}$. By Corollary (17.0) above, this means $g(z) = \lambda$ is constant and $|\lambda| = |g(z_0)| = 1$. In other words $f(z) = \lambda z$ ( since $g(z) = \frac{f(z)}{z}$ for $z \neq 0$).

---

* Hermann Schwarz 25/1/1843 – 30/11/1921

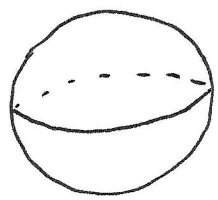
(17.2) Schwarz' Lemma is used in studying behavior of holomorphic functions from $\mathbb{D} \longrightarrow \mathbb{D}$ $(\mathbb{D} = D(0;1))$. The unit disc $\mathbb{D}$ plays an important role in the theory of Riemann surfaces, conformal geometry and hyperbolic geometry – because of the following two foundational results of Riemann.

**Riemann mapping theorem.** Let $\Omega \subset \mathbb{C}$ be a proper, open, connected and simply connected subset of $\mathbb{C}$ (proper : $\Omega \neq \mathbb{C}$). Then

$\Omega \cong \mathbb{D}$ (meaning $\exists$ holomorphic $f : \mathbb{D} \to \Omega$ ; $g : \Omega \to \mathbb{D}$ s.t. $f \circ g = Id_{\Omega}$ ; $g \circ f = Id_{\mathbb{D}}$).

**Riemann uniformization theorem.** Up to conformal equivalence, there are only 3 simply connected Riemann surfaces :

Riemann Sphere

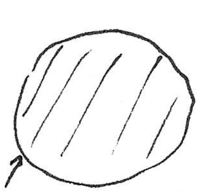


denoted by $\hat{\mathbb{C}}$, (or $\mathbb{P}^1(\mathbb{C})$); (or $S^2$)

Complex plane

denoted by $\mathbb{C}$.

Unit disc



denoted by $\mathbb{D}$.

boundary not included

We will prove Riemann mapping theorem later in this course, but Riemann uniformization theorem is beyond our scope (since we are not going to discuss the general theory of Riemann Surfaces).

(17.3) <u>Using Schwarz' Lemma to determine automorphisms of $\mathbb{D}$.</u>

A <u>conformal automorphism</u> (also sometimes called <u>holomorphic reparametrization</u>) of $\mathbb{D}$ is a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ which admits a holomorphic inverse (i.e., $\exists\, g : \mathbb{D} \to \mathbb{D}$ holomorphic s.t. $f \circ g = g \circ f = Id_{\mathbb{D}}$, i.e. $f(g(z)) = g(f(z)) = z$).

Let $Aut(\mathbb{D})$ denote the set (or group) of all conformal automorphisms of $\mathbb{D}$.

- rotations. – for each $\theta \in \mathbb{R}$, let $R_\theta : \mathbb{D} \to \mathbb{D}$ be given by

$$R_\theta(z) = e^{i\theta} \cdot z \quad \text{(rotation by } \theta\text{)}.$$

Lemma. – If $f \in Aut(\mathbb{D})$ is such that $f(0) = 0$, then there exists $\theta \in \mathbb{R}$ so that $f = R_\theta$.

Proof. By Schwarz' Lemma, $|f(z)| \leq |z|$. If $g \in Aut(\mathbb{D})$ is the inverse of $f$, then $g(0) = 0$ and again by Schwarz' Lemma $|g(w)| \leq |w|$. Setting $w = f(z)$ we get

$$|z| = |g(f(z))| \leq |f(z)|$$

Hence $|f(z)| = |z| \ \forall z \in \mathbb{D}$, and again applying Schwarz' Lemma

We get that $\exists\, \lambda \in \mathbb{C}$, $|\lambda| = 1$ s.t. $f(z) = \lambda z$.
(as $|\lambda| = 1$, $\lambda = e^{i\theta}$ for $\theta = \arg(\lambda)$). $\square$

For $\alpha \in \mathbb{D}$, let $T_\alpha : \mathbb{D} \to \mathbb{C}$ be given by

$$T_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}$$

(Note: $T_\alpha$ is defined on $\mathbb{C} - \{\frac{1}{\bar{\alpha}}\}$. As $\left|\frac{1}{\bar{\alpha}}\right| = \frac{1}{|\alpha|} > 1$, this set contains $\mathbb{D}$).

Proposition.- (1) For each $\alpha \in \mathbb{D}$, $T_\alpha \in \mathrm{Aut}(\mathbb{D})$.

(2) Let $f \in \mathrm{Aut}(\mathbb{D})$. Then there exists $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that
$$f(z) = e^{i\theta}\, \frac{z - \alpha}{1 - \bar{\alpha} z} \quad \forall\, z \in \mathbb{D}$$
(i.e. $f = R_\theta \circ T_\alpha$).

Proof.- (1) Note that $T_\alpha$ is defined and continuous on $\overline{\mathbb{D}}$ $(= \{|z| \leq 1\})$.
On the boundary, i.e. for $z = e^{i\varphi}$ $(0 \leq \varphi \leq 2\pi)$ we have

$$|T_\alpha(e^{i\varphi})| = \frac{|e^{i\varphi} - \alpha|}{|1 - e^{i\varphi}\bar{\alpha}|} = \frac{|e^{i\varphi} - \alpha|}{|e^{i\varphi}| \cdot |e^{-i\varphi} - \bar{\alpha}|}$$

$$= \frac{|e^{i\varphi} - \alpha|}{|\overline{e^{i\varphi} - \alpha}|} = 1.$$

Thus, by maximum modulus principle (Cor. 16.4), $|T_\alpha(z)| < 1$ $\forall z \in \mathbb{D}$.
(it cannot be $= 1$ on any point of $\mathbb{D}$, since in that case $T_\alpha$ would be

a constant - and we know from its formula that $T_\alpha$ is not a constant). Hence $T_\alpha(\mathbb{D}) \subset \mathbb{D}$ (i.e. $T_\alpha : \mathbb{D} \to \mathbb{D}$).

Claim: $T_\alpha(T_{-\alpha}(z)) = z$. This proves that $T_\alpha \in Aut(\mathbb{D})$.

(Pf. of the claim: $T_\alpha(T_{-\alpha}(z)) = T_\alpha\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right) = \dfrac{\frac{z+\alpha}{1+\bar{\alpha}z} - \alpha}{1-\bar{\alpha}\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right)}$

$$= \frac{z+\alpha-\alpha-|\alpha|^2 z}{1+\bar{\alpha}z-\bar{\alpha}z-|\alpha|^2} = \frac{(1-|\alpha|^2)z}{1-|\alpha|^2} = z. \;)$$

(2) Now let $f \in Aut(\mathbb{D})$. Let $\beta = f(0)$, and $g = T_\beta \circ f$.

Then $g(0) = T_\beta(\beta) = 0$, $g \in Aut(\mathbb{D}) \Rightarrow \exists\, \theta \in \mathbb{R}$ s.t. $g = R_\theta$.

(see Lemma on page 4)

So $T_\beta \circ f = R_\theta \Rightarrow f = T_{-\beta} \circ R_\theta$.

Claim: $T_w \circ R_\theta = R_\theta \circ T_{e^{-i\theta}(w)}$.

( Pf: $T_w(R_\theta(z)) = T_w(e^{i\theta}z) = \dfrac{e^{i\theta}z - w}{1-e^{i\theta}z\bar{w}} = e^{i\theta}\left(\dfrac{z-e^{-i\theta}w}{1-z\overline{(e^{-i\theta}w)}}\right)$

$$= R_\theta\left(T_{e^{-i\theta}w}(z)\right) \qquad \square \;)$$

Thus $f = T_{-\beta} \circ R_\theta = R_\theta \circ T_{-e^{-i\theta}\beta}$ $\square$