Lecture 18

(18.0) Recall. last time we studied <u>conformal automorphisms</u> of the unit disc $\mathbb{D} = D(0;1) = \{z : |z| < 1\}$.

$\mathrm{Aut}(\mathbb{D})$ (automorphisms of the unit disc) $= \{ f : \mathbb{D} \to \mathbb{D}$ holomorphic such that there exists $g : \mathbb{D} \to \mathbb{D}$ also holomorphic with $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{D}} \}$

(i.e. $f(g(z)) = z = g(f(z)) \ \forall z \in \mathbb{D}$)

We proved that any $f \in \mathrm{Aut}(\mathbb{D})$ has the following form: $\exists\, \theta \in \mathbb{R}, \alpha \in \mathbb{D}$:

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

Let $R_\theta \in \mathrm{Aut}(\mathbb{D})$ $(\theta \in \mathbb{R})$ be given by $R_\theta(z) = e^{i\theta} z$ (rotation by angle θ)

and T_α $(\alpha \in \mathbb{D})$ be given by $T_\alpha(z) = \dfrac{z - \alpha}{1 - \bar{\alpha} z}$. "translation". Then

$$\mathrm{Aut}(\mathbb{D}) = \left\{ R_\theta \circ T_\alpha : \begin{array}{l} \theta \in \mathbb{R} \ (\mathrm{mod}\ 2\pi\mathbb{Z}) \\ \alpha \in \mathbb{D} \end{array} \right\}.$$

(18.1)* Some algebraic properties of R_θ, T_α:

(i) $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1 + \theta_2}$ is obvious

(ii) $R_\theta T_\alpha R_{\theta^{-1}} = T_{R_\theta(\alpha)}$

Proof $R_\theta\left(T_\alpha\left(R_{\theta^{-1}}(z)\right)\right) = e^{i\theta} \dfrac{e^{-i\theta} z - \alpha}{1 - \bar{\alpha} e^{-i\theta} z} = \dfrac{z - e^{i\theta}\alpha}{1 - \overline{(e^{i\theta}\alpha)}\, z}$

$$= T_{e^{i\theta}\alpha}(z) = T_{R_\theta(\alpha)}(z) \qquad \square$$

*Optional

(iii) $T_\alpha \circ T_\beta = R_\varphi \circ T_\gamma$ where $\varphi = \arg\left(\dfrac{1+\alpha\bar\beta}{1+\bar\alpha\beta}\right)$ and

$$\gamma = T_{-\beta}(\alpha) = \frac{\alpha+\beta}{1+\bar\beta\alpha}.$$

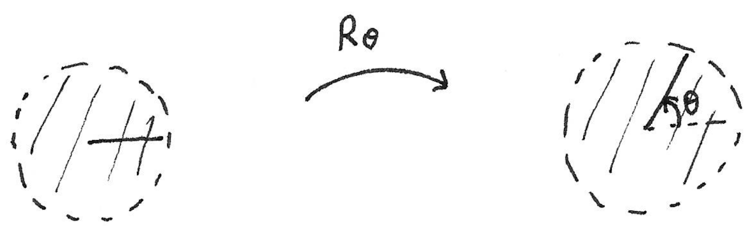
Proof:-
$$T_\alpha(T_\beta(z)) = T_\alpha\left(\frac{z-\beta}{1-\bar\beta z}\right) = \frac{\dfrac{z-\beta}{1-\bar\beta z} - \alpha}{1-\bar\alpha\left(\dfrac{z-\beta}{1-\bar\beta z}\right)}$$

$$= \frac{z-\beta-\alpha+\bar\beta\alpha z}{1-\bar\beta z-\bar\alpha z+\bar\alpha\beta} = \left(\frac{1+\alpha\bar\beta}{1+\bar\alpha\beta}\right)\left(\frac{z-\dfrac{\alpha+\beta}{1+\alpha\bar\beta}}{1-\overline{\left(\dfrac{\alpha+\beta}{1+\alpha\bar\beta}\right)}z}\right) \quad \alpha\bar\beta$$

$$= e^{i\varphi}\cdot T_{\frac{\alpha+\beta}{1+\alpha\bar\beta}}(z) \qquad \square$$

(18.2)* To get some geometric intuition about these automorphisms, we can try to draw some lines/circles in z-plane and see their image in $w = f(z)$-plane; where $f \in \operatorname{Aut}(\mathbb{D})$.

The case when $f = R_\theta$ is very easy - it is just a rotation.



In particular it preserves angles and distances (in the Euclidean sense).

The following picture makes it clear that T_α cannot possibly preserve Euclidean distance.

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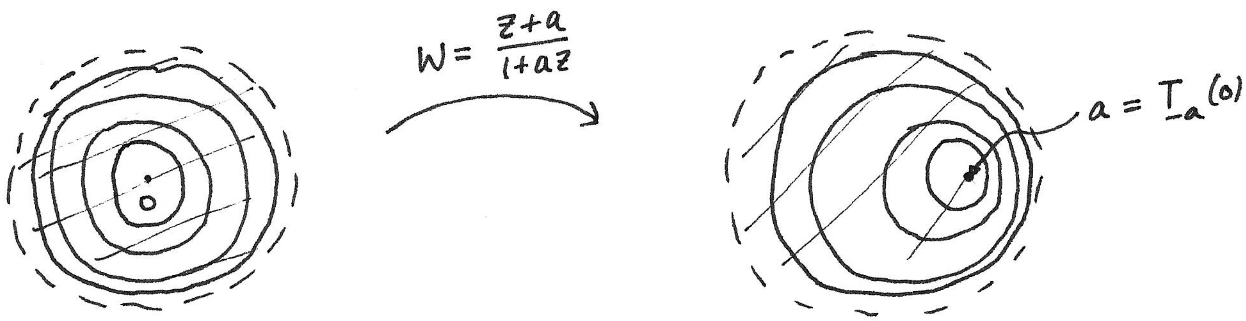
*Optional

Let us assume $a \in \mathbb{R}$; $0 \le a < 1$ - for simplicity and consider $z \longmapsto \dfrac{z+a}{1+az} = T_{-a}(z)$ sending 0 to a.



Some circles centered at 0

Exercise: image of a circle $C(0;r)$ in z-plane ; under T_{-a} , is again a circle centered at $\dfrac{a(1-r^2)}{1-a^2r^2}$; of radius $\dfrac{r(1-a^2)}{1-a^2r^2}$.

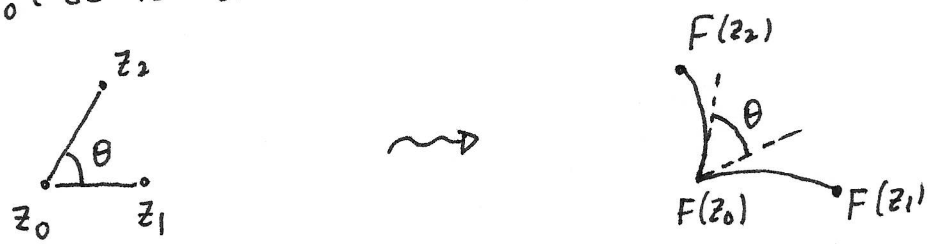
(18.3)* More properties of T_α and invariant arc length.

(1) $$\frac{d}{dz}\left(T_\alpha(z)\right) = \frac{d}{dz}\left(\frac{z-\alpha}{1-\bar\alpha z}\right) = \frac{1}{1-\bar\alpha z} + \frac{\bar\alpha(z-\alpha)}{(1-\bar\alpha z)^2}$$

$$= \frac{1 - \bar\alpha z + \bar\alpha z - |\alpha|^2}{(1-\bar\alpha z)^2}$$

$$T_\alpha'(z) = \frac{1-|\alpha|^2}{(1-\bar\alpha z)^2}$$ does not vanish (for any $z \in \mathbb{D}$)

Remark.- If $F : \Omega \to \mathbb{C}$ ($\Omega \subset \mathbb{C}$ open) is holomorphic ; and if $z_0 \in \Omega$ is such that $F'(z_0) \neq 0$; then <u>F preserves angles</u> at z_0.



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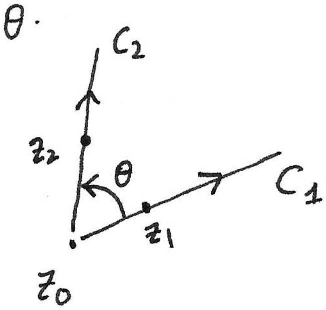
* Optional

The term "conformal mapping" is used to signify preservation of angles. We will return to this topic later in the course.

(Sketch of a proof of the remark made above: - Let C_1 and C_2 be two rays emanating from z_0, at an angle θ.

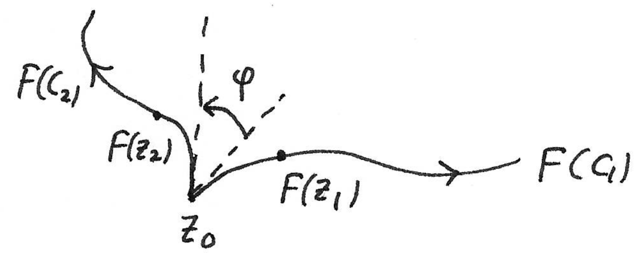


That is, for any z_1 on C_1 and z_2 on C_2,

$$\arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \theta .$$

Let $\gamma_1 = F(C_1)$; $\gamma_2 = F(C_2)$.

The angle (from γ_1 to γ_2), say φ, can be defined as :



$$\varphi = \arg\left(\frac{F(z_2)-F(z_0)}{F(z_1)-F(z_0)}\right) \quad \text{as } z_1, z_2 \to z_0 .$$

i.e.
$$\varphi = \lim_{\substack{z_1 \to z_0 \\ z_2 \to z_0}} \arg \left(\frac{\dfrac{F(z_2)-F(z_0)}{z_2 - z_0}}{\dfrac{F(z_1)-F(z_0)}{z_1 - z_0}} \cdot \frac{z_2 - z_0}{z_1 - z_0} \right)$$

$$= \arg \frac{F'(z_0)}{F'(z_0)} \cdot \frac{z_2 - z_0}{z_1 - z_0} = \theta . \qquad \square \;)$$

(2) $$1 - |T_\alpha(z)|^2 = 1 - \frac{|z-\alpha|^2}{|1-\bar{\alpha} z|^2} = \frac{|1-\bar{\alpha} z|^2 - |z-\alpha|^2}{|1-\bar{\alpha} z|^2}$$

$$= \frac{1 + |\alpha|^2|z|^2 - \bar{\alpha} z - \alpha\bar{z} - |z|^2 - |\alpha|^2 + \bar{\alpha} z + \alpha \bar{z}}{|1-\bar{\alpha} z|^2}$$

(recall: $|z_1 - z_2|^2$
$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$
$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$)

$$= \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\bar{\alpha}z|^2}$$

Combining (1) and (2) we get $\left(\begin{array}{l} (1) \quad T_\alpha'(z) = \dfrac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2} \\ (2) \quad 1-|T_\alpha(z)|^2 = \dfrac{(1-|\alpha|^2)(1-|z|^2)}{|1-\bar{\alpha}z|^2} \end{array} \right)$

$$|T_\alpha'(z)| = \frac{1-|T_\alpha(z)|^2}{1-|z|^2}$$

i.e. $$\boxed{\frac{|T_\alpha'(z)|}{1-|T_\alpha(z)|^2} = \frac{1}{1-|z|^2}}$$

(18.4)* Hyperbolic distance. The formula discovered above allows us to define a new "arc length" - which is invariant under $\mathrm{Aut}(\mathbb{D})$.

Namely, given a path $\gamma : [0,1] \longrightarrow \mathbb{D}$, define:

$$\ell_H(\gamma) = \int_0^1 \frac{|\gamma'(t)|\, dt}{1-|\gamma(t)|^2} \qquad \text{(hyperbolic arc length)}$$

The notion of "arc length" leads to the definition of "distance" between two points

$$d_H(\alpha,\beta) = \inf \{ \ell_H(\gamma) : \gamma \text{ path joining } \alpha \text{ and } \beta \}$$

$(\alpha, \beta \in \mathbb{D})$

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* Optional