Lecture 19

(19.0) Sequences and series of complex numbers.

Recall - given a sequence $\{z_n\}_{n=1}^{\infty}$, we say $\boxed{\lim_{n\to\infty} z_n = \ell}$ if for every $\varepsilon > 0$, we can find N such that $|z_n - \ell| < \varepsilon$ for all $n \geq N$.

Cauchy's criterion. - For $\{z_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} z_n$ exists if and only if for every $\varepsilon > 0$, there exists N s.t. $|z_n - z_m| < \varepsilon$ for all $n, m \geq N$.

Given a series $\sum_{n=0}^{\infty} u_n$, we form a sequence of partial sums $S_n = u_0 + u_1 + \dots + u_n$ $(n \geq 0)$. The convergence or divergence of $\sum_{n=0}^{\infty} u_n$ is then the same for the sequence $\{S_n\}_{n=0}^{\infty}$.

$$\sum_{k=0}^{\infty} u_k := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\sum_{k=0}^{n} u_k \right)$$

(19.1) Ratio and root tests for absolute convergence.

A series $\sum_{n=0}^{\infty} u_n$ is said to be absolutely convergent if $\sum_{n=0}^{\infty} |u_n|$ is convergent. By triangle inequality, it is easy to see that

$$\text{Absolute Convergence} \Rightarrow \text{Convergence}$$

but the converse does not hold (Ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but not absolutely.)

<u>d'Alembert's Ratio Test</u> (1768). Consider $\sum_{n=0}^{\infty} u_n$ and assume $u_n \neq 0$. ($\forall n \geq 0$)

- If $\limsup_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, then $\sum_{n=0}^{\infty} u_n$ converges absolutely.
- If $\liminf_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$ then $\sum_{n=0}^{\infty} u_n$ diverges.

<u>Cauchy's root test</u> (1821). Let $\rho = \limsup_{n\to\infty} (|u_n|)^{1/n}$. If $\rho < 1$, $\sum_{n=0}^{\infty} u_n$ converges absolutely. If $\rho > 1$, the series diverges.

(recall $\ell = \limsup_{n\to\infty} x_n$ means that for every $\ell_1 < \ell < \ell_2$;
$\{n : x_n > \ell_1\}$ is infinite; $\{n : x_n > \ell_2\}$ is finite.)

<u>Proof of Root test</u>: (i) If $\rho < 1$, let us choose $\rho_2 \in (\rho, 1)$. Then, by definition of lim sup, $\exists N$ s.t. $|u_n|^{1/n} < \beta \quad \forall n \geq N$.

$$\Rightarrow \sum_{n=N}^{\infty} |u_n| < \sum_{n=N}^{\infty} \beta^n = \frac{\beta^N}{1-\beta}.$$

(ii) Note that by Cauchy's Criterion for convergence,

$$\sum_{n=0}^{\infty} u_n \text{ converges} \iff \forall \varepsilon > 0, \exists N \text{ s.t. } \left|\sum_{k=n}^{m} u_k\right| < \varepsilon$$
for every $m \geq n \geq N$.

In particular $\sum_{n=0}^{\infty} u_n$ converges $\Rightarrow \lim_{n\to\infty} u_n = 0$. (Take $m = n$)

Thus, if $\limsup_{n\to\infty} |u_n|^{1/n} = \rho > 1$, we have $|u_n| > 1$ for infinitely many n's, hence $\lim_{n\to\infty} u_n$ cannot be 0. $\square$

(19.2) Examples. (i) $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent. (e.g. by ratio test).

(ii) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Proof left as an exercise).

(iii) $\sum_{n=1}^{\infty} \frac{1}{n^s}$ $(s \in \mathbb{R};\ s>1)$ is convergent.

(Proof. $\frac{1}{2^s} + \frac{1}{3^s} < \frac{2}{2^s} = \frac{1}{2^{s-1}}$.

$\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} < \frac{4}{4^s} = \frac{4}{4^{s-1}}$. Continuing this way, we get the sum of first $2^p - 1$ terms

$$\sum_{n=1}^{2^p-1} \frac{1}{n^s} < \frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \dots + \frac{1}{2^{(p-1)(p-1)}} < \frac{1}{1-2^{1-s}}.$$

$\square$)

If $z \in \mathbb{C}$, and $\frac{1}{n^z} = \frac{1}{n^x} \cdot \frac{1}{n^{iy}}$ (as usual $n^z = e^{z\ln(n)}$)

Then $|n^z| = n^{Re(z)}$. Hence, by (iii), $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely for $Re(z) > 1$.

(19.3) Sequences of functions. Let $\Omega \subset \mathbb{C}$ be an open and connected set. Let $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of functions. We say $\{f_n\}_{n=1}^{\infty}$ <u>converges pointwise</u> if for every $z \in \Omega$, the sequence of complex numbers $\{f_n(z)\}_{n=1}^{\infty}$ converges.

In this case, we can define $f : \Omega \to \mathbb{C}$ by $f(z) = \lim_{n \to \infty} f_n(z)$.

Given a subset $A \subset \Omega$, we say $\{f_n\}_{n=1}^{\infty}$ converges <u>uniformly relative to A</u> if for every $\varepsilon > 0$, we can find $N > 0$ s.t.

$$|f_n(z) - f_m(z)| < \varepsilon, \quad \forall\ n, m \geq N, \quad z \in A.$$

(19.4) Remark. Unfolding the definition of convergence, saying that $\{f_n\}_{n=1}^{\infty}$ converges pointwise means; given $z \in \Omega$ and $\varepsilon > 0$, we can find $N > 0$ s.t. $|f_n(z) - f_m(z)| < \varepsilon$, $\forall\ n, m \geq N$.

Uniform convergence (rel. to A) signifies that this N can be chosen to work for every $z \in A$.

The notion of uniform convergence is due to Weierstrass, and was introduced to preserve useful properties of functions – e.g. continuity. These properties are known to fail if we only know pointwise convergence. Meaning, $f_n \to f$ pointwise; each f_n continuous, does not imply that f will be continuous.

Example. $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$ $(x \in \mathbb{R})$. Pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \frac{1}{2} & \text{if } x = \pm 1 \\ 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| > 1 \end{cases}.$$

(19.5) [$\Omega \subset \mathbb{C}$ open, connected. $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ sequence of functions]

We say $\{f_n\}_{n=1}^{\infty}$ converges uniformly <u>relative to compact sets</u>; if given $\underset{\text{cpct}}{K \subset \mathbb{C}}$; $K \subset \Omega$; $\{f_n\}_{n=1}^{\infty}$ converges uniformly rel. to K.

i.e. given $K \subset \Omega$ (K: cpct) and $\varepsilon > 0$, we can find $N > 0$ s.t.

$$|f_n(z) - f_m(z)| < \varepsilon \qquad \forall\ n, m \geq N \text{ and } z \in K.$$

(This N depends only on ε and K).

Since a singleton $\{z_0\}$ is compact, uniform convergence rel. to compact sets implies pointwise convergence – and again we can define $f : \Omega \to \mathbb{C}$ as the pointwise limit $\quad f(z) := \lim_{n\to\infty} f_n(z)$.

(19.6) <u>Theorem (Weierstrass*)</u>. Let $\Omega \subset \mathbb{C}$ be an open, connected set, $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ be a uniformly convergent (rel. to compact sets) sequence of functions and $f : \Omega \to \mathbb{C}$ the pointwise limit of $\{f_n\}_{n=1}^{\infty}$.

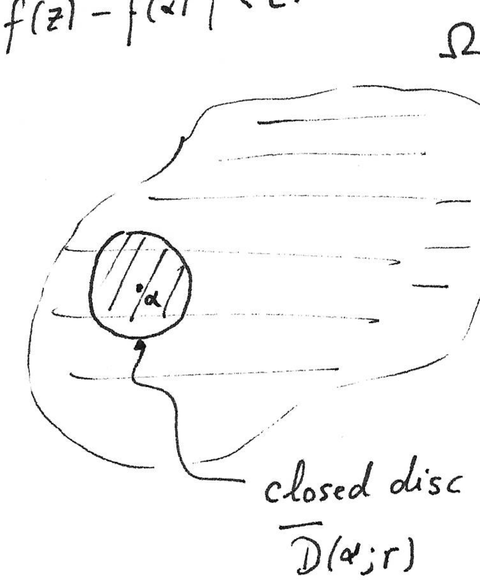
(1) If f_n is continuous for all n, then so is f.

(2) If $\gamma : [a,b] \to \Omega$ is a piecewise smooth curve, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz \qquad \left(\begin{array}{l}\text{assuming continuity of } f_n\text{'s} \\ \text{and } f, \text{ as in (1)}\end{array}\right)$$

(3) If f_n is holomorphic $\forall n$; then so is f. Moreover $\{f_n'\}_{n=1}^{\infty}$ converges uniformly (rel. to compact sets) to f'.

Proof. (1) We have to prove that given $\alpha \in \Omega$ and $\varepsilon > 0$, we can find $\delta > 0$ such that $|z - \alpha| < \delta \Rightarrow |f(z) - f(\alpha)| < \varepsilon$.



- Fix $r > 0$ s.t. $\overline{D}(\alpha;r) \subset \Omega$.
- By uniform convergence, $\exists N$ s.t.
  $$|f_n(z) - f(z)| < \varepsilon/3$$
  for every $n \geq N$ and $z \in \overline{D}(\alpha;r)$.
- By continuity of f_N, $\exists \tilde{\delta} > 0$ s.t.
  $$|z - \alpha| < \tilde{\delta} \Rightarrow |f_N(z) - f_N(\alpha)| < \frac{\varepsilon}{3}$$

Take $\delta < \text{Min}(r, \tilde{\delta})$. Then we get :

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*Karl Weierstrass (31/10/1815 – 19/2/1897)

$$|f(z)-f(\alpha)| = |f(z)-f_N(z)+f_N(z)-f_N(\alpha)+f_N(\alpha)-f(\alpha)|$$
$$\leq |f(z)-f_N(z)|+|f_N(z)-f_N(\alpha)|+|f_N(\alpha)-f(\alpha)|$$
$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

(2) Assuming continuity of $\{f_n\}$ and hence that of f (as in (1)), let $\gamma : [a,b] \to \Omega$ be a piecewise smooth path. For our compact set, take $K = \{\gamma(t) : a \leq t \leq b\}$. Let $L = \text{length}(\gamma)$. Then, by uniform continuity, $\exists\, N > 0$ s.t.

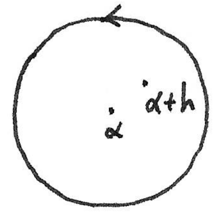
$$|f_n(z) - f(z)| < \frac{\varepsilon}{L}, \quad \forall\, z \text{ on } K;\ n \geq N.$$

Hence, $\forall\, n \geq N$, we get

$$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| = \left| \int_\gamma (f_n(z) - f(z))\,dz \right|$$
$$\leq \int_\gamma |f_n(z) - f(z)|\,dz < \frac{\varepsilon}{L}\cdot L = \varepsilon.$$

(3). We begin by showing that f is $\mathbb{C}$-differentiable at every $\alpha \in \Omega$ and $f'(\alpha) = \frac{1}{2\pi i}\int_{C(\alpha;r)} \frac{f(z)}{(z-\alpha)^2}\,dz$. Choose $r>0$ s.t. $\overline{D}(\alpha;r) \subset \Omega$ and let $C(\alpha;r)$ = circle (counter-clockwise) of radius r centered at α.

For $h \in \mathbb{C}$, $|h| < r$, we have



$C(\alpha;r)$

$$\frac{1}{h}\left(f(\alpha+h) - f(\alpha)\right) = \lim_{n\to\infty} \frac{1}{h}\left(f_n(\alpha+h) - f_n(\alpha)\right)$$

$$= \lim_{n\to\infty} \frac{1}{2\pi i h} \int_{C(\alpha;r)} \left(\frac{f_n(z)}{z-\alpha-h} - \frac{f_n(z)}{z-\alpha}\right) dz$$ (by Cauchy's integral formula) [each f_n is holomorphic]

$$= \lim_{n\to\infty} \frac{1}{2\pi i \cancel{h}} \int_{C(\alpha;r)} \frac{f_n(z)\,\cancel{h}}{(z-\alpha-h)(z-\alpha)}\, dz$$

Thus, we have to prove that $$\lim_{n\to\infty} \int_{C(\alpha;r)} \frac{f_n(z)}{(z-\alpha-h)(z-\alpha)}\, dz = \int_{C(\alpha;r)} \frac{f(z)}{(z-\alpha-h)(z-\alpha)}\, dz \quad \text{— (A)}$$

and $$\lim_{h\to 0} \int_{C(\alpha;r)} \frac{f(z)}{(z-\alpha-h)(z-\alpha)}\, dz = \int_{C(\alpha;r)} \frac{f(z)}{(z-\alpha)^2}\, dz \quad \text{— (B)}.$$

The proof of (B) is the same as given in Lecture 13, page 5; and of (A) follows along the same lines as in (2). Namely, given $\varepsilon > 0$, choose $N > 0$ s.t. $|f_n(z) - f(z)| < \frac{\varepsilon}{2\pi}(\cancel{R}\ r - |h|)$. Then $(\forall\, n \geq N;\ z \in C(\alpha;r))$

$$\left| \int_{C(\alpha;r)} \frac{f_n(z) - f(z)}{(z-\alpha-h)(z-\alpha)} \cdot dz \right| \leq \frac{\varepsilon}{2\pi}(r-|h|) \cdot \frac{1}{(r-|h|)\cdot r} \cdot 2\pi r = \varepsilon.$$

bound for $|f_n(z) - f(z)|$ $\quad$ $|z-\alpha-h| \geq |z-\alpha| - |h| = r - |h|$

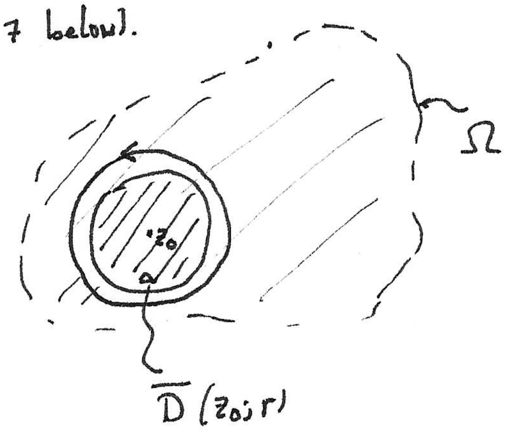
Hence, we proved that f is holomorphic. The same argument as in (2) above shows that

$$\underbrace{\lim_{n\to\infty} \frac{1}{2\pi i}\int_{C(\alpha;r)} \frac{f_n(z)}{(z-\alpha)^2}\,dz}_{=f_n'(\alpha)} = \underbrace{\frac{1}{2\pi i}\int_{C(\alpha;r)} \frac{f(z)}{(z-\alpha)^2}\,dz}_{=f'(\alpha).}$$

i.e. $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to f'.

Finally, to prove that $f_n' \to f$ is uniform (rel. to compact sets) it suffices to prove it for $K = \overline{D}(z_0;r) \subset \Omega$ (see Lemma 19.7 below).

Since Ω is open and distance $(K;\partial\Omega) > 0$,
we can choose $R > r$ s.t. $\overline{D}(z_0;R) \subset \Omega$.



Given $\varepsilon > 0$, choose $N > 0$ s.t.

$$\cancel{|f_n(z) - f(z)| < \varepsilon' \ \forall\, n \geq N,\ z \in \overline{D}(z} \quad \cancel{\left(\varepsilon' = \frac{(R-r)^2}{R}\varepsilon\right)}$$

$$|f_n(z) - f(z)| < \frac{\varepsilon \cdot (R-r)^2}{R} \quad \forall\, n \geq N \text{ and } \forall\, z \in C(z_0;R).$$

Then
$$|f_n'(w) - f'(w)| = \left| \frac{1}{2\pi i} \int_{C(z_0;R)} \frac{f_n(z) - f(z)}{(z-w)^2}\,dw \right|$$

$$< \frac{1}{2\pi}\,\frac{\varepsilon}{R}(R-r)^2\,\frac{1}{(R-r)^2}\cdot 2\pi R = \varepsilon$$

(since for $z \in C(z_0;R)$, $w \in \overline{D}(z_0;r)$, $|z-w| \geq R-r$)

for every $w \in \overline{D}(z_0;r)$ and $n \geq N$.

$\square$

(19.7) **Lemma.** A sequence $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ of functions converges uniformly relative to compact sets $\iff$ it converges uniformly relative to closed discs.

Proof. Since each closed disc is compact, the forward implication follows.

For the converse, let $K \subset \Omega$ be an arbitrary compact set.

Since Ω is open, given any $z \in \Omega$, we can find $r_z \in \mathbb{R}_{>0}$ s.t. $\overline{D}(z;r_z) \subset \Omega$. Thus we get an open cover $K \subset \bigcup_{z \in K} D(z,r_z)$,

and compactness of K implies that there exist $z_1, \dots, z_m \in K$ s.t.

$(r_j = r_{z_j})$ $\qquad K \subset \bigcup_{j=1}^{m} D(z_j;r_j)$.

Now let $\varepsilon > 0$ be given. Using uniform convergence relative to closed discs, we can find N_j $(1 \le j \le m)$ s.t. $|f_n(z) - f_m(z)| < \varepsilon$, $\forall\, n, m \ge N_j$; $\forall\, z \in \overline{D}(z_j;r_j)$.

Take $N = \mathrm{Max}\{N_1, \dots, N_m\}$ which works for all $z \in K$. $\square$