Lecture 20

(20.0) Recall - last time we proved a very important theorem due to Weierstrass. Given $\Omega \subset \mathbb{C}$ (open, connected) and a sequence $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ of functions, converging uniformly relative to compact sets, to $f : \Omega \to \mathbb{C}$ (meaning: for every compact set $K \subset \Omega$ and $\varepsilon > 0$, we can find $N > 0$ such that $|f_n(z) - f(z)| < \varepsilon \ \forall n \geq N \ \& \ z \in K$.)

we have:

(1) f_n is continuous $\forall n \Rightarrow f$ is continuous. In this case

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz \quad \text{for any piecewise smooth } \gamma : [a,b] \to \Omega.$$

(2) f_n is holomorphic $\forall n \Rightarrow f$ is holomorphic. Moreover, $\{f_n'\}_{n=1}^{\infty}$ converges uniformly (rel. to compact sets) to f'.

i.e. $\displaystyle \lim_{n\to\infty} f_n' = f'$.

Remark.- (1) is true over $\mathbb{R}$ as well, but (2) is not. For example consider $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$. Each f_n is differentiable, but $\displaystyle\lim_{n\to\infty} f_n(x) = |x|$ is not.

Even more, by Weierstrass' approximation theorem, any continuous function on $[0,1]$ can be obtained as a uniform limit of polynomials.

(20.1) **Power Series.** – A <u>power series</u> is a special kind of series of functions : $A(z) = \sum_{n=0}^{\infty} a_n z^n$; where $a_0, a_1, \ldots \in \mathbb{C}$.

Thus sequence of partial sums of $A(z)$ are polynomials :

$$\left\{ S_N(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_N z^N \right\}_{N=0}^{\infty},$$ which are holomorphic functions $S_N : \mathbb{C} \to \mathbb{C}$. The derivative of S_N is given by :

$$S_N'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + N a_N z^{N-1} \quad (N \geq 1)$$
$$S_0'(z) = 0.$$

(20.2) **Abel's Theorem*** (radius of convergence of a power series).

Given a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, there is a unique $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that <u>$A(z)$ converges uniformly (rel. to compact sets) on $D(0;R)$</u> and if $|z| > R$, $A(z)$ diverges. Moreover, for $|z|<R$, the convergence of $A(z)$ is <u>absolute</u>.

Proof.– Consider the set $I = \{ r \in \mathbb{R}_{\geq 0} :$ there exists $M \in \mathbb{R}_{>0}$ so that $|a_n| \cdot r^n < M \ \forall n \}$.

(same argument as in Cauchy's root test).

- $I \neq \phi$ $(0 \in I)$.
- I is an interal. Clearly if $t \in I$ and $s<t$, then $s \in I$.

Take $R = \sup(I)$. Note : R could be ∞ if $I = [0,\infty)$.

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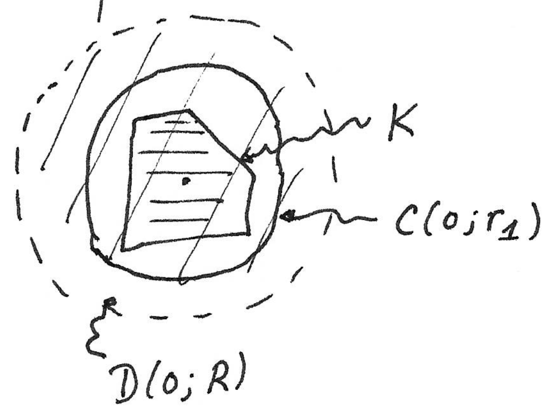
<u>If $|z| > R$, then $A(z)$ diverges.</u> By construction, if $|z| > R$, then $\{|a_n| \cdot |z|^n\}_{n=0}^{\infty}$ is an unbounded sequence. In particular $\lim_{n \to \infty} a_n z^n \neq 0$. Hence $\sum_{n=0}^{\infty} a_n z^n$ is divergent.

<u>$A(z)$ converges uniformly on $D(0;R)$</u> (rel. to compact sets) : If $R = 0$, $D(0;R) = \phi$ and there is nothing to prove.

Assume $R > 0$. Let $K \subset D(0;R)$ be a compact set.

As K is bounded, there exists $r_1 < R$ s.t. $|z| \leq r_1 \ \forall z \in K$.



Choose $r_2 \in (r_1, R)$. By construction, there exists a constant $M \in \mathbb{R}_{>0}$ s.t.

$$|a_n| \cdot r_2^n < M \quad \text{for every } n.$$

Thus, for $z \in K$, we have

$$\left| \sum_{n=0}^{N} a_n z^n \right| \leq \sum_{n=0}^{N} |a_n| \cdot |z|^n \quad (\text{by triangle inequality})$$

$$< \sum_{n=0}^{N} \frac{M}{r_2^n} \cdot r_1^n = M \cdot \frac{1 - t^{N+1}}{1 - t} \qquad \left(t = \frac{r_1}{r_2} < 1\right)$$

(sum of geometric series)

$$< \frac{M}{1-t}.$$

Hence $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly. In more explicit terms and absolutely.

given $\varepsilon > 0$, choose N large enough so that $t^{N+1} \dfrac{M}{1-t} < \varepsilon$.

Then $\forall\, n \geq N$ and $p \geq 0$, the modulus of the partial sum

$$|S_{n+p}(z) - S_n(z)| = \left| \sum_{k=n+1}^{n+p} a_k z^k \right| \leq \sum_{k=n+1}^{n+p} |a_k| |z|^k$$

$$\leq \sum_{k=n+1}^{n+p} \frac{M}{r_2^k} r_1^k < t^{n+1} \sum_{\ell=0}^{\infty} M \cdot t^\ell < \frac{t^{N+1} \cdot M}{1-t} < \varepsilon.$$

$\square$

(for every $z \in K$)

(20.3) Remarks. - (1) Combining Abel's and Weierstrass' Theorems, we get that (if $R > 0$), $A(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function $D(0; R) \to \mathbb{C}$, being a uniform limit of polynomial functions $\left\{ S_N(z) = \sum_{n=0}^{N} a_n z^n \right\}_{N=0}^{\infty}$. Moreover

$$A'(z) = \lim_{N \to \infty} S_N'(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1},$$

i.e. power series can be termwise differentiated within its disc of convergence.

(2) $\sum_{n=0}^{\infty} a_n z^n$ is often called power series centered at 0. A bit more generally, power series centered at $\alpha \in \mathbb{C}$ is of the form $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$.

(3) In practice we compute the radius of convergence of a given power series using ratio / root tests.

Root test formula (Hadamard) : $\boxed{\dfrac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}}$.

Ratio test is a bit more restrictive. Assuming $a_n \neq 0$ $\forall n$ and $\ell = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists, we get $R = \dfrac{1}{\ell}$. ($R=\infty$ if $\ell = 0$).

(20.4) Examples. - (1) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Using ratio test,

$$\lim_{n\to\infty}\left(\frac{1}{(n+1)!}\cdot n!\right) = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$ Hence its radius of convergence is ∞.

(2) $\displaystyle\sum_{n=0}^{\infty} z^n$ has radius of convergence $= 1$.

(3) $\displaystyle\sum_{n=0}^{\infty} n^k z^n$ ($k \in \mathbb{Z}_{\geq 0}$) has radius of convergence $= 1$.

(4) $\displaystyle\sum_{n=0}^{\infty} n!\, z^n$ has radius of convergence $= 0$.

(5) (Exercise) If $R > 0$ is the radius of convergence of $A(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$, then R is also the radius of convergence of $A'(z) = \displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ and

$$\sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$ (termwise antiderivative of $A(z)$).

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \; ; \quad R_1$$
$$B(z) = \sum_{n=0}^{\infty} b_n z^n \; ; \quad R_2$$
Radii of conv. — assume $R_1, R_2 > 0$.

(20.5) Algebraic operations on power series.

(1) Addition :

$$A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$ has radius of convergence $\geq \mathrm{Min}\{R_1, R_2\}$.

(Proof. - if $t < \mathrm{Min}\{R_1, R_2\}$, then $\exists\, M_1, M_2 \in \mathbb{R}_{>0}$ s.t.

$|a_n|\, t^n < M_1$
$|b_n|\, t^n < M_2$ $\quad \forall n$. Hence $|a_n + b_n| \cdot t^n \leq (|a_n| + |b_n|)\, t^n < M_1 + M_2$.

By the construction given in the proof of Abel's Theorem - we get that the radius of convergence of $A(z) + B(z)$ is atleast $\mathrm{Min}\{R_1, R_2\}$ □)

(2) Multiplication :

$$A(z) \cdot B(z) = \sum_{n=0}^{\infty} \left(a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0\right) z^n.$$

radius of convergence $\geq \mathrm{Min}\{R_1, R_2\}$.

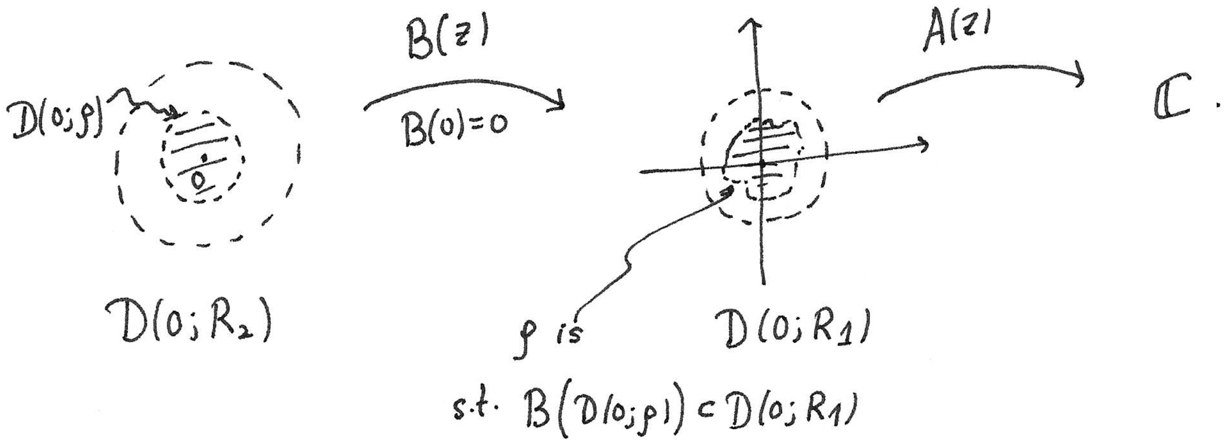
(Proof. - Let $|z| = r < \mathrm{Min}\{R_1, R_2\}$. Then $\sum_{k=0}^{\infty} |a_k|\, r^k = \alpha$ and $\sum_{l=0}^{\infty} |b_l|\, r^l = \beta$ are convergent series. Now it is easy to see that $\left|(a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)\right| \cdot r^n < \alpha \cdot \beta \quad \forall n$ □)

(3) Composition. Assume $b_0 = 0$ (i.e., $B(0) = 0$) and define $C(z) = A(B(z)) = \sum_{n=0}^{\infty} a_n B(z)^n$ (composition of A and B).

As $\sum_{k=1}^{\infty} |b_k| r^k \to 0$ as $r \to 0$; we can find $\rho > 0$ s.t.

$$\forall r < \rho : \quad \sum_{k=1}^{\infty} |b_k| r^k < R_1 \quad (\text{radius of convergence of } A(z))$$

Hence radius of convergence of $C(z)$ is $\geq \rho$ is non-zero.



(4) Exercise. Use (3) to prove that - if $A(z) = a_0 + a_1 z + \cdots$ has non-zero radius of convergence and $a_0 \neq 0$; then the algebraic inverse of $A(z)$ (i.e. a power series $C(z)$ s.t. $A(z) \cdot C(z) = 1$) exists and has non-zero radius of convergence.