Lecture 21

(21.0) Recall that in the previous two lectures we proved two useful theorems.

- (Weierstrass) If $f_n \to f$ uniformly (rel. to compact sets) (here, $f_n, f : \Omega \to \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$) then

  (1) Continuity of f_n, $\forall n$, implies continuity of f, in which case
  $$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz \quad (\gamma : [a,b] \to \Omega \text{ piecewise smooth})$$

  (2) Holomorphicity of f_n $\forall n$, implies that of f, in which case $f_n' \to f'$ uniformly rel. to compact sets.

- (Abel) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series ($a_n \in \mathbb{C}$ $\forall n \in \mathbb{Z}_{\geq 0}$).

  Then there is a unique $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that:

  (1) The sequence of partial sums $\left\{ S_N(z) = \sum_{n=0}^{N} a_n z^n \right\}_{N=0}^{\infty}$ converges uniformly (rel. to compact sets) and absolutely on $D(0;R)$.

  (2) If $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ is divergent.

Hence $A(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function
$$A : D(0;R) \to \mathbb{C}$$
which can be termwise differentiated and integrated:
$$A'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1} : D(0;R) \to \mathbb{C}$$
an antiderivative of A.

The coefficients $a_0, a_1, a_2, \dots$ of $A(z)$ can be recovered from the holomorphic function $D(0;R) \longrightarrow \mathbb{C}$ it defines, as follows:

$$a_0 = A(0), \quad a_1 = A'(0), \quad \text{and for any } m \geq 0,$$

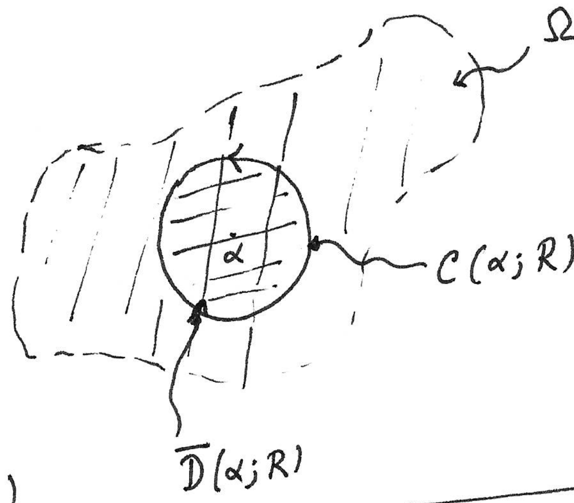
$$A^{(m)}(z) = \left(\frac{d}{dz}\right)^m (A(z)) = \sum_{n=m}^{\infty} (n+m)(n+m-1)\cdots(n+1)\, a_{n+m} z^n$$

$$= \sum_{n=m}^{\infty} \frac{(n+m)\cdots(n+1)\cdot n(n-1)\cdots 1}{n(n-1)\cdots 1} a_{n+m} z^n = \sum_{n=m}^{\infty} \frac{(n+m)!}{n!} z^n a_{n+m}$$

Hence $A^{(m)}(0) = m!\, a_m$ i.e. $\boxed{a_m = \dfrac{A^{(m)}(0)}{m!}}$.

(21.1) Taylor Series expansion of a holomorphic function.

Conversely, assume that $f : \Omega \to \mathbb{C}$ is a holomorphic function, where $\Omega \subset \mathbb{C}$ is an open set. Let $\alpha \in \mathbb{C}$ and let $R \in \mathbb{R}_{>0}$ be such that $\overline{D}(\alpha; R) \subset \Omega$.



Theorem.* There is a unique power series centered at α, of radius of convergence $\geq R$, $F(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ s.t.

$$f(z) = F(z) \quad \text{for every } z \in D(\alpha; R).$$

The coefficients $c_0, c_1, \dots$ are given by: $\boxed{c_n = \dfrac{f^{(n)}(\alpha)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{C(\alpha;R)} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz}$

* Brook Taylor (1685 – 1731) obtained this formal series in his book Methodus incrementorum dated 1715. The theorem as stated here, and its proof are due to Cauchy – Cours d'Analyse 1822.

Proof. - Define $c_n := \dfrac{f^{(n)}(\alpha)}{n!}$ $(n \in \mathbb{Z}_{\geq 0})$ and consider the power series $F(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$. We will now show that the radius of convergence of $F(z)$ is at least R. So, let $r \in [0, R)$. As in the proof of Abel's Theorem, we will show that there is a constant $M \in \mathbb{R}_{>0}$ s.t. $|c_n| \cdot r^n < M$ $\forall\, n \in \mathbb{Z}_{\geq 0}$.

Let $C_r = C(\alpha; r)$ be the counterclockwise circle of radius r, centered at α, and take $M > \mathrm{Max}\{|f(z)| : z \in C_r\}$. By Cauchy's integral formula

$$c_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

$$\Rightarrow \quad |c_n| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz \right| < \frac{1}{2\pi} \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{M}{r^n}$$

$\Rightarrow |c_n| \cdot r^n < M$ for every n, as claimed.

Hence the power series $F(z)$ defines a holomorphic function

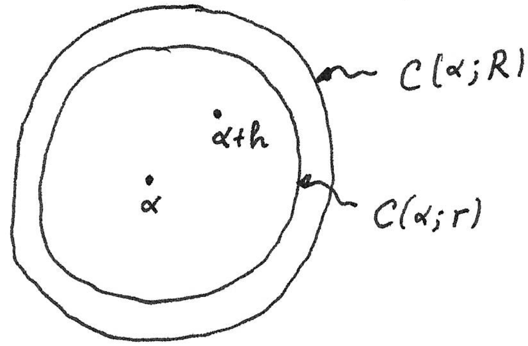
$$F : D(\alpha; R) \to \mathbb{C}.$$

We will now prove that $F(w) = f(w)$ $\forall\, w \in D(\alpha; R)$.

To see this, let us write $w = \alpha + h$ (so $|h| < R$).

Choose r to lie between $|h|$ and R $(|h| < r < R)$.

and as before let $C_r = C(\alpha; r)$.



Again by Cauchy's integral formula

$$f(\alpha+h) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-\alpha-h}\, dz.$$

For $z \in C_r$; $|z-\alpha| = r > |h|$. Hence we have

$$\frac{1}{z-\alpha-h} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{h}{z-\alpha}} = \frac{1}{z-\alpha} \sum_{n=0}^{\infty} \frac{h^n}{(z-\alpha)^n} \qquad (*)$$

$$\Rightarrow \quad f(\alpha+h) = \frac{1}{2\pi i} \int_{C_r} \left(\sum_{n=0}^{\infty} h^n \cdot \frac{f(z)}{(z-\alpha)^{n+1}} \right) dz$$

$$= \sum_{n=0}^{\infty} h^n \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz \right)$$

[valid by uniform convergence of geometric series $(*)$]

$$= \sum_{n=0}^{\infty} h^n \frac{f^{(n)}(\alpha)}{n!}$$

$$= \sum_{n=0}^{\infty} c_n (w-\alpha)^n = F(w) \qquad (\text{recall: } w = \alpha + h).$$

$\square$

(21.2) Examples. (1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

Since for $f(z) = e^z$, we have $f^{(n)}(z) = e^z \ \forall n \geq 0$, we get the coefficients of the Taylor Series $C_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$. Hence

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{on } D(0;\infty) = \mathbb{C}.$$

(2) $$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad (\forall z \in \mathbb{C}).$$

Method 1: for $f(z) = \sin(z)$, $f'(z) = \cos(z)$, $f''(z) = -\sin(z)$, ...

i.e. $$\begin{aligned} f^{(2k)}(z) &= (-1)^k \sin(z) \\ f^{(2k+1)}(z) &= (-1)^k \cos(z) \end{aligned} \implies \begin{aligned} f^{(2k)}(0) &= 0 \\ f^{(2k+1)}(0) &= (-1)^k. \end{aligned}$$

Method 2: $$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(1-(-1)^n)}{2i} \frac{(iz)^n}{n!} \quad \text{(from Example (1))}$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{2}}{\cancel{2}\cancel{i}} \, i^{2k+\cancel{1}} \cdot \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$(n = 2k+1)$

(3) Take $\frac{d}{dz}$ of (2): $$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

for every $z \in \mathbb{C}$.

(4) Geometric Series.

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n \quad \forall z \in D(0;1)$$

Note : while the left-hand size is defined $\forall z \in \mathbb{C} - \{1\}$, this identity only makes sense within the disc of convergence of $\sum_{n=0}^{\infty} z^n$; i.e. $D(0;1)$.

In the proof of Taylor's theorem, for $f(z) = \frac{1}{1-z}$ and $\alpha = 0$, we can only take $R < 1$ since f is not defined at $z = 1$.

(5) Taking antiderivative of (4) we get

$$-\log(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \quad \forall z \in D(0;1)$$

(Note: $\log(1) = 0$; so the choice of constant disappears)

or,

$$\log(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1}$$

or,

$$\log(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} \quad \forall z \in D(1;1)$$

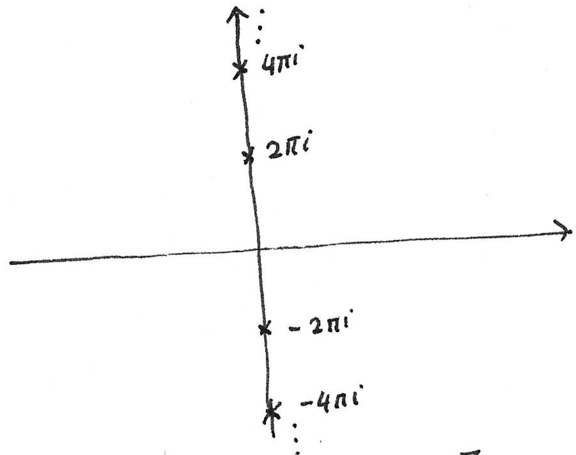
(21.3) Bernoulli Numbers. Let $f(z) = \frac{z}{e^z - 1}$. Note that

$f(0) = 1$ $\left(\lim_{z \to 0} \frac{z}{e^z - 1} = 1 \right)$. The nearest (to 0) $\alpha \in \mathbb{C}$

for which $f(\alpha)$ is not defined is $\alpha = \pm 2\pi i$. Since the domain of f is $\mathbb{C} - \{z \neq 0 : e^z = 1\}$ i.e. $\mathbb{C} - 2\pi i \mathbb{Z}_{\neq 0}$

Consider the Taylor Series expansion of f near 0.

[its radius of convergence $= 2\pi$].



$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Domain of $f(z) = \dfrac{z}{e^z - 1}$

- $$f(-z) = \frac{-z}{e^{-z} - 1} = \frac{-z e^z}{1 - e^z} = \frac{z e^z}{e^z - 1}$$

$$\Rightarrow \quad f(-z) - f(z) = \frac{z(e^z - 1)}{e^z - 1} = z.$$

i.e. $\displaystyle\sum_{n=0}^{\infty} \left((-1)^n c_n - c_n\right) z^n = z$. Thus $c_1 = \frac{-1}{2}$; $c_3 = c_5 = \cdots = 0$.

We already know that $c_0 = 1$.

Bernoulli numbers can be defined as $\boxed{\dfrac{z}{e^z - 1} = \displaystyle\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}}$

Thus $B_0 = 1$, $B_1 = \frac{-1}{2}$; $B_{2k+1} = 0 \quad \forall\, k \geq 1$.

<u>Ex.</u> $\displaystyle\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$ (recursive formula for Bernoulli numbers)