Lecture 22

(22.0) Recall we showed:

- Given a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, we obtain a holomorphic function $D(0;R) \to \mathbb{C}$ (R = radius of convergence of $A(z)$, assumed to be non-zero here).
- Conversely, given a holomorphic function $f : \Omega \to \mathbb{C}$ (Ω: open) and $\alpha \in \Omega$, we get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n \quad \forall\, z \in D(\alpha;R) \text{ where } R>0 \text{ is s.t. } D(\alpha;R) \subset \Omega.$$

Cauchy's estimates. for any $r \in [0,R)$, let $M(r) = \mathrm{Max}\{|f(z)| : z \in C(\alpha;r)\}$

Then $\boxed{\left|\frac{f^{(n)}(\alpha)}{n!}\right| \leq \frac{M(r)}{r^n}}$ $\forall\, n \geq 0$. (see Lecture 21 – page 3)

Remark. In Problem 18 of Set 4, we see that if $\left|\frac{f^{(n)}(\alpha)}{n!}\right| = \frac{M(r)}{r^n}$ (for some $n \geq 0$ and $r \in (0,R)$) then $f(z) = c_n z^n$ $\left(c_n = \frac{f^{(n)}(\alpha)}{n!}\right)$.

(22.1) Some remarks on the behaviour of a power series on the circle of convergence.

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with non-zero radius of convergence. By scaling the variable z, we may assume that the radius of convergence is 1.

Theorem (Abel). - Assume $\sum_{n=0}^{\infty} a_n$ is convergent. Then

$$\lim_{\substack{r \to 1 \\ (0<r<1)}} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} a_n$$

Proof. - Let $S_N = a_0 + a_1 + \dots + a_N$. $(N \geq 0)$. Then $a_n = S_n - S_{n-1}$.

We are given that $S = \lim_{N \to \infty} S_N$ exists. Note: (assuming $S_{-1} = 0$).

$$\sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{N} (S_n - S_{n-1}) z^n = (1-z) \sum_{n=0}^{N} S_n z^n + S_{N+1} z^{N+1}$$

(hence, for $|z|<1$, $\sum_{n=0}^{\infty} a_n z^n = (1-z) \sum_{n=0}^{\infty} S_n z^n$)

To prove: $\lim_{\substack{r \to 1 \\ (0<r<1)}} \left| \sum_{n=0}^{\infty} a_n r^n - S \right| = 0.$

Given $\varepsilon > 0$, choose $M > 0$ s.t. $|S_n - S| < \frac{\varepsilon}{2}$ for every $n \geq M$.

We split the sum $\sum_{n=0}^{\infty} a_n r^n - S = \left(\sum_{n=0}^{\infty} S_n r^n \right)(1-r) - (1-r) \cdot S \cdot (1-r)^{-1}$

$$= (1-r) \sum_{n=0}^{\infty} (S_n - S) r^n$$

$$= \underbrace{(1-r) \sum_{n=0}^{M} (S_n - S) r^n}_{\mathrm{I}} + \underbrace{(1-r) \sum_{n=M+1}^{\infty} (S_n - S) r^n}_{\mathrm{II}}$$

Note that the second term has modulus $\leq (1-r) \cdot \frac{\varepsilon}{2} \cdot \sum_{n=M+1}^{\infty} r^n < \frac{\varepsilon}{2}$.

and the first term $\to 0$ as $r \to 1$.

So, for the first term, if $C = \sum_{n=0}^{M} |s_n - s|$, then for $r \in (0,1)$ s.t. $|1-r| < \frac{\varepsilon}{2C}$, we get

$$\left|\sum_{n=0}^{\infty} a_n r^n - s\right| \leq \left|(1-r)\sum_{n=0}^{M}(s_n - s) r^n\right| + \left|(1-r)\sum_{n=M+1}^{\infty}(s_n-s) r^n\right|$$

$$< \frac{\varepsilon}{2C}\cdot C + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

(22.2) Abel's Theorem can be applied to evaluate certain infinite series in terms of known functions. In order to do so, we have to check first that the series is convergent

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$.

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent by Alternating Series test (a special case of Dirichlet's test - see Problem 6 of Set 4).

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1}}{n} = \log(1+z)$ for $|z| < 1$.

By Abel's Theorem $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \lim_{\substack{r\to 1\\(0<r<1)}} \log(1+r) = \log(2)$.

Counter example. - $\displaystyle\sum_{n=0}^{\infty} z^n (-1)^n = \frac{1}{1+z}$ for $|z| < 1$.

for $z = 1$: $\quad \displaystyle\lim_{\substack{r \to 1 \\ (0<r<1)}} \sum_{n=0}^{\infty} (-1)^n r^n = \frac{1}{2}$ exists

but $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ is divergent.

(22.3) By simple rotation, we can conclude from Abel's Theorem, that if $\displaystyle\sum_{n=0}^{\infty} a_n \zeta^n$ is convergent for some $\zeta \in C(0;1)$ (i.e. $|\zeta| = 1$) then

$$\lim_{\substack{z \to \zeta \\ \frac{z}{\zeta} \in (0,1)}} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \zeta^n$$

(22.4)* In general, problem of determining subsets of $C(0;1)$ where $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges can have (almost) any answer.

Examples: (1) $\displaystyle\sum_{n=1}^{\infty} n z^n$ diverges for every $z \in C(0;1)$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges for every $z \in C(0;1)$.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for every $z \in C(0;1) \sim \{1\}$
diverges for $z = 1$.

(4) $\displaystyle\sum_{n=1}^{\infty} z^{n!}$ diverges on $E = \{ z \in C(0;1) : \exists\, l \in \mathbb{Z}_{\geq 1} \text{ s.t. } z^l = 1 \} \underset{\text{dense}}{\subset} C(0;1)$. (Problem 9, Set 4)

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* Optional

I recommend "The behavior of power series on their circle of convergence" by T.W. Körner, for more details of the problem of determining which sets can arise as the set of points where a power series can converge.

(22.5) Order of vanishing. Assume $f : \Omega \to \mathbb{C}$ is a holomorphic function defined on an open and connected set Ω. Let $\alpha \in \Omega$ be such that $f(\alpha) = 0$. Consider the Taylor series expansion of f near α:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \qquad (z \in D(\alpha; R))$$

Note: $c_0 = 0$. We have two possibilities:

(i) $c_n = 0 \quad \forall\, n \geq 1$.

(ii) $c_0 = c_1 = \cdots = c_{N-1} = 0$; $c_N \neq 0$. (here $N \geq 1$).

In the second case, we say <u>f vanishes at α to order N</u>, or <u>the order of vanishing of f at α</u> is N. We can write

$$f(z) = (z-\alpha)^N \left(c_N + c_{N+1}(z-\alpha) + \cdots \right)$$
$$= (z-\alpha)^N g(z)$$

where $g(z) = \sum_{\ell=0}^{\infty} c_{N+\ell} (z-\alpha)^{\ell}$ also converges on $D(\alpha; R)$

Note : $g(\alpha) = C_N \neq 0$. Hence we can find $\rho > 0$ such that

$$z \in D(\alpha; \rho) \Rightarrow g(z) \neq 0.$$

Thus $f(z) = (z-\alpha)^N g(z) \neq 0$ for every $z \in D(\alpha;\rho)$ $z \neq \alpha$.

We record these observations as the following lemma.

<u>Lemma.</u> $[f : \Omega \to \mathbb{C}$ holomorphic ; $\alpha \in \Omega]$

If $f(\alpha) = 0$ then either $f \equiv 0$ on $D(\alpha; R)$

or there exists $0 < \rho < R$ s.t. $f(z) \neq 0$ $\forall\, z \in D(\alpha;\rho) - \{\alpha\}$.

In other words, <u>zeroes of a holomorphic function</u> (not identically zero near a point) <u>are isolated</u>.

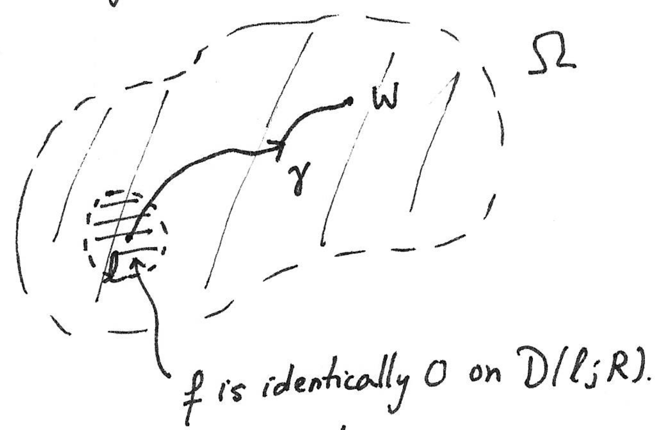
(22.6) Identity Theorem. – Let $\Omega \subset \mathbb{C}$ be an open and connected set and $f_1, f_2 : \Omega \to \mathbb{C}$ be two holomorphic functions. Assume that there is a convergent sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ converging to a point $\lim_{n\to\infty} z_n = \ell \in \Omega$; such that

$$f_1(z_n) = f_2(z_n) \quad \forall\, n \geq 1.$$

Then $f_1(z) = f_2(z)$ $\forall\, z \in \Omega$.

(This theorem was advertised in Lecture 16 - § 16.3).

Proof.- Let $f = f_1 - f_2$. Thus $f(z_n) = 0 \ \forall n \geq 1$ and by continuity $f(\ell) = 0$. As a consequence of Lemma 22.5 (last page) - since ℓ is not an isolated zero of f, $\exists\, R > 0$ s.t. $f(z) = 0 \ \forall z \in D(\ell; R)$.



Now let $w \in \Omega$ and let $\gamma : [0,1] \to \Omega$ be a continuous path joining ℓ to w.

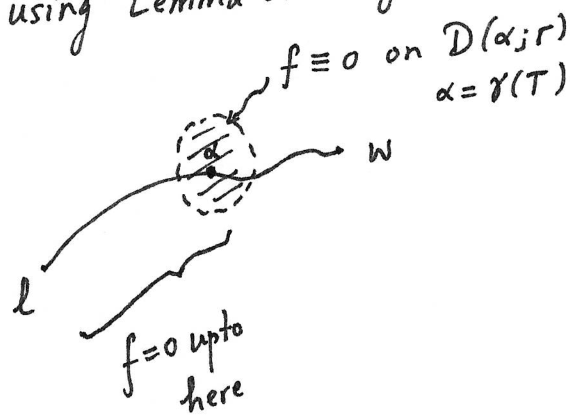
Define
$$I = \left\{ t \in [0,1] \text{ s.t. } \begin{array}{c} f(\gamma(s)) = 0 \\ \forall\, 0 \leq s \leq t \end{array} \right\}$$

Note : $0 \in I$ (hence $I = \phi$) ; $t \in I$ and $t' < t \Rightarrow t' \in I$ (hence I is an interval).

Claim : $I = [0,1]$. This implies $f(w) = f(\gamma(1)) = 0$.

Proof. Let $T = \operatorname{Sup}(I)$. Note that by continuity of f, $f(\gamma(T)) = 0$ showing that $T \in I$ (i.e. I is a closed interval).

If $T < 1$, then as $\alpha = \gamma(T)$ is not an isolated zero of f, $\exists\, r > 0$ s.t. $f(z) = 0 \ \forall z \in D(\alpha; r)$ (using Lemma 22.5 again)



As γ is continuous, we can find $\varepsilon > 0$ s.t. $\gamma([T, T+\varepsilon)) \subset D(\alpha; r)$

Contradicting the fact that $T = \sup(I)$.

$\square$