Lecture 23

(23.0) Recall that we have been discussing applications of Taylor series expansion of holomorphic functions. Last time we proved that:

- Zeroes of non-constant holomorphic functions are isolated. Meaning, if $f:\Omega\to\mathbb{C}$ is a holomorphic function, and $f(\alpha)=0$ for some $\alpha\in\Omega$, then

  either f is identically zero in a disc around α

  or $f(z)\neq 0$ for every z from a disc around α ($z\neq\alpha$).

- If $f_1, f_2:\Omega\to\mathbb{C}$ are two holomorphic functions defined on an open and connected set $\Omega\subset\mathbb{C}$, and assume that $\exists\,\{z_n\}_{n=1}^{\infty}$, ℓ in Ω such that $\lim_{n\to\infty} z_n=\ell$; $f_1(z_n)=f_2(z_n)$ $\forall n$.

  Then $f_1(z)=f_2(z)$ for every $z\in\Omega$.

(23.1) Inverse of a power series.

<u>Prop.</u> - Let $A(z)=\sum_{n=0}^{\infty} a_n z^n$ be a power series with non-zero radius of convergence. Assume that <u>$A(0)=0$ and $A'(0)\neq 0$.</u>

(i.e. $A(z) = a_1 z + a_2 z^2 + \dots$; $a_1\neq 0$)

Then there exists (unique) $B(z)=\sum_{n=1}^{\infty} b_n z^n$ such that

$$A(B(z)) = z$$

Moreover the radius of convergence of $B(z)$ is also non-zero

Proof. - The existence and uniqueness of $B(z)$ is purely an algebraic statement. Let $B(z) = b_1 z + b_2 z^2 + \dots$ where the coefficients $b_1, b_2, \dots$ are to be determined from the condition that $A(B(z)) = z$.

$$z = A(B(z)) = a_1 B(z) + a_2 B(z)^2 + a_3 B(z)^3 + \dots$$

$$= a_1 (b_1 z + b_2 z^2 + b_3 z^3 + \dots) + a_2 (b_1 z + b_2 z^2 + \dots)^2 + a_3 (b_1 z + b_2 z^2 + \dots)^3 + \dots$$

Comparing coefficients of z, we get $a_1 b_1 = 1 \Rightarrow b_1 = \frac{1}{a_1}$ (recall $a_1 \neq 0$)

_____ z^2, we get $a_1 b_2 + a_2 b_1^2 = 0$

_____ z^3, we get $a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3 = 0$

Continuing, we get $\forall n \geq 2$;
$$a_1 b_n + \sum_{l=2}^{n} a_l \cdot \underbrace{\text{Coeff. of } z^n \text{ in } (b_1 z + b_2 z^2 + \dots)^l}_{\text{only involves } b_1, \dots, b_{n-1}} = 0$$

Hence, $\{b_1, b_2, \dots\}$ are uniquely determined. (if and only if $a_1 \neq 0$).

Aside: a bit more explicitly,

Coefficient of z^n in $B(z)^l = (b_1 z + b_2 z^2 + b_3 z^3 + \dots)^l$

$$= \sum_{\substack{K_1, K_2, \dots \in \mathbb{Z}_{\geq 0} \\ \text{s.t. } K_1 + K_2 + \dots + \dots = l \\ K_1 + 2K_2 + 3K_3 + \dots = n}} \frac{l!}{K_1! \, K_2! \cdots} b_1^{K_1} b_2^{K_2} \cdots$$

} Note: if $l \geq 2$; $K_n = K_{n+1} = \dots = 0$

(Note: this sum is finite)

Let $P_{n;l}(\underline{b})$ = coeff. of z^n in $B(z)^l$ = polynomial in $b_1, b_2 \dots, b_{n-1}$ with coefficients from $\mathbb{Z}_{\geq 0}$.

In conclusion, $\forall\, n \geq 2$, we have

$$a_1 b_n = -\sum_{\ell=2}^{n} a_\ell \, P_{n;\ell}(b_1, b_2, \dots b_{n-1}) \qquad - \quad (*)$$

<u>Convergence of $B(z)$</u>. Convergence of an algebraically constructed power series is often investigated using the trick of "majorizing series".

Let $R \in \mathbb{R}_{>0} \cup \{\infty\}$ be the radius of convergence of $A(z)$.

Assume (for simplicity) that $a_1 = 1$.

Let $r < R$ and let $M > 0$ be such that $|a_n| < \frac{M}{r^n}$ $(\forall\, n \geq 2)$

Define $\tilde{A}(X) = X - \sum_{\ell=2}^{\infty} \frac{M}{r^\ell} \cdot X^\ell$ and $\tilde{B}(X) = \sum_{n=1}^{\infty} \beta_n X^n$

be its inverse (constructed as above).

Note: $\beta_1 = 1$ and $\beta_n = \sum_{\ell=2}^{n} \frac{M}{r^\ell} \underbrace{P_{n;\ell}(\beta_1, \dots, \beta_{n-1})}_{\text{polynomial with coefficients from } \mathbb{Z}_{\geq 0}}$

$\Rightarrow \quad \beta_n \in \mathbb{R}_{\geq 0}$.

Moreover $\left(\text{since } |a_\ell| < \frac{M}{r^\ell}\right)$, we have $|b_n| \leq \beta_n$ $(\forall\, n \geq 1)$.

Thus to prove that $\sum_{n=1}^{\infty} b_n z^n$ has non-zero radius of convergence, it suffices to show that $\sum_{n=1}^{\infty} \beta_n X^n$ has non-zero radius of convergence

Let $Y = \tilde{B}(X)$. Since $\tilde{A}(X) = X - \sum_{\ell=2}^{\infty} \frac{M}{r^\ell} X^\ell$

$$= X - \frac{M \cdot X^2}{r^2} \cdot \frac{1}{1 - \frac{X}{r}} \qquad (\text{for } |X| < r)$$

$$\tilde{A}(X) = X - \frac{MX^2}{r(r-X)}.$$ The equation $\tilde{A}(\tilde{B}(X)) = X$

becomes a quadratic equation for $Y = \tilde{B}(X)$:

$$Y - \frac{MY^2}{r(r-Y)} = X.$$ That is, Y is the unique soln. ($=0$ for $X=0$)

of: $(M+r)Y^2 - r(X+r)Y + r^2 X = 0$

$$\Rightarrow Y = \frac{r(X+r)}{2(M+r)}\left(1 - \left(1 - \frac{4X(M+r)}{(X+r)^2}\right)^{1/2}\right).$$

Using $$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n$$ for $|z| < 1$ (see §23.2 below)

we get $Y = \tilde{B}(X)$ as a power series with non-zero radius of convergence. □

(23.2) Binomial Theorem for power series.

For $\alpha \in \mathbb{C}$, $(1+z)^\alpha = \exp(\alpha \log(1+z))$ defined on

$\{z \in \mathbb{C} : 1+z \notin \mathbb{R}_{\leq 0}\} = \mathbb{C} - \mathbb{R}_{\leq -1}$.

Power Series expansion

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n$$ for $|z| < 1$.

[if $\alpha \in \mathbb{Z}_{\geq 0}$, the series written here is a polynomial - so radius of convergence $= \infty$.]

$$\frac{d}{dz}(1+z)^\alpha = \alpha \cdot (1+z)^{\alpha-1} = \frac{\alpha}{1+z}(1+z)^\alpha .$$

Writing $(1+z)^\alpha = \sum_{n=0}^{\infty} c_n z^n$ $\quad (c_0 = 1)$ we get

$$\sum_{n=1}^{\infty} n \cdot c_n \cdot z^{n-1} = \alpha \cdot \left(\sum_{n=0}^{\infty} (-1)^n z^n \right) \left(\sum_{n=0}^{\infty} c_n z^n \right)$$

Coefficient of z^0 : $\quad c_1 = \alpha \cdot 1 \cdot c_0 = \alpha$.

" " z^1 : $\quad 2 \cdot c_2 = \alpha (c_1 - c_0) = \alpha(\alpha - 1) \Rightarrow c_2 = \frac{\alpha(\alpha-1)}{2}$

" " z^2 : $\quad 3 c_3 = \alpha (c_2 - c_1 + c_0)$

$$= \alpha \left(\frac{\alpha(\alpha-1)}{2} - \alpha + 1 \right) = \frac{\alpha(\alpha-1)(\alpha-2)}{2}$$

$$\Rightarrow c_3 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!}$$

Continuing we get (coeff of z^{n-1})

$$n c_n = \alpha \left(c_{n-1} - c_{n-2} + \cdots + (-1)^{n-1} c_0 \right)$$

$$= \alpha c_{n-1} - \alpha \left(c_{n-2} - c_{n-3} + \cdots + (-1)^n c_0 \right)$$

$$= \alpha c_{n-1} - \alpha \cdot (n-1) c_{n-1}$$

$$= c_{n-1} (\alpha - n + 1)$$

Hence $c_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. $\square$

(23.3) $f(z) = \sin^{-1}(z)$. $= \sum_{n=0}^{\infty} c_n z^n$ $\quad (c_0 = 0)$ is uniquely determined by $z = \sin(\sin^{-1}(z))$. To compute this series, however, it is easier to take the derivative of $\sin(\sin^{-1} z) = z$ to get

$$\cos(\sin^{-1}(z)) \cdot \frac{d}{dz}(\sin^{-1} z) = 1 \quad \Rightarrow \quad \frac{d}{dz}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz}\sin^{-1}(z) = (1-z^2)^{-\frac{1}{2}} = \sum_{n \geq 0} (-1)^n \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} z^{2n} \quad (|z|<1)$$

$$= \sum_{n=0}^{\infty} \frac{1(3)\cdots(2n-1)}{n! \cdot 2^n} z^{2n}$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n}}{2^{2n}}$$

$$\left(1 \cdot 3 \cdots (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n)!}{2^n \, n!} \right)$$

Hence $\sin^{-1}(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{2^{2n} \cdot (2n+1)}$

for $|z| < 1$.