Lecture 24

(24.0) Last time we proved the analogue of "inverse function theorem" for power series.

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series such that $A(0) = 0$ and $A'(0) \neq 0$, then there is a unique power series $B(z) = \sum_{n=1}^{\infty} b_n z^n$ s.t. $A(B(z)) = z$.

If A has a non-zero radius of convergence then so does B.

Carrying out the conclusion of this statement for B (instead of A) we obtain a unique power series $C(z)$ (with non-zero radius of convergence s.t. $B(C(z)) = z$. It is a standard argument to show that $A = C$:

$$A = A \circ Id = A \circ (B \circ C) = (A \circ B) \circ C = Id \circ C = C.$$

In conclusion, by shrinking the radius of convergence of A if necessary, we have: $A : D(0; \rho_1) \to D(0; \rho_2)$ and $B : D(0; \rho_2) \to D(0; \rho_1)$ are inverse to each other.

(24.1) In terms of holomorphic functions, if $f : \Omega \to \mathbb{C}$ is holomorphic and $\alpha \in \Omega$ is such that $f'(\alpha) \neq 0$, then $\exists\, \rho_1, \rho_2 > 0$ s.t. $f : D(\alpha; \rho_1) \to D(f(\alpha); \rho_2)$ has a holomorphic inverse.

(24.2) Behavior near critical points (i.e. $f'(\alpha)=0$).

Example. Let $m \in \mathbb{Z}_{\geq 2}$ and let $f(z) = z^m$. Thus $f(0)=0$ is a zero of order m. Moreover, for any $w \in \mathbb{C}$, $w \neq 0$, the equation $f(z) = w$ has exactly m distinct solutions. The general case is very much similar to this situation.

Let $\Omega \subset \mathbb{C}$ be an open set; $f : \Omega \to \mathbb{C}$ a holomorphic function; $\alpha \in \Omega$, and assume that

$$f'(\alpha) = f''(\alpha) = \cdots = f^{(N-1)}(\alpha) = 0 \ ; \ f^{(N)}(\alpha) \neq 0$$

i.e. we are assuming that f is not constant near α; and $N \in \mathbb{Z}_{\geq 1}$ is the smallest positive integer for which $f^{(N)}(\alpha) \neq 0$.

In other words, Taylor Series expansion of f, near α, has the form:

$$f(z) = f(\alpha) + \sum_{n=N}^{\infty} c_n (z-\alpha)^n \qquad (z \in D(\alpha; R))$$

$\left[R \in \mathbb{R}_{>0} \text{ is s.t. } D(\alpha; R) \subset \Omega. \right]$

Proposition. – There exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$f : D(\alpha; \rho_1) \longrightarrow D(f(\alpha); \rho_2)$$

is <u>N-to-1</u> on $D^{\times}(\alpha; \rho_1)$, i.e. $\forall\, w \in D(f(\alpha); \rho_2)$, $w \neq f(\alpha)$, there exist N distinct $z_1, z_2, \ldots, z_N \in D^{\times}(\alpha; \rho_1) = D(\alpha;\rho_1) - \{\alpha\}$ (punctured disc) with $f(z_1) = \cdots = f(z_N) = w$.

Proof. - For simplicity: take $\alpha = 0$ and $f(\alpha) = 0$ so that the Taylor Series expansion of f near 0 has the form:

$$f(z) = C_N z^N + C_{N+1} z^{N+1} + \dots$$

Moreover, let us assume that $C_N = 1$.

(These are not serious restrictions. I am simply replacing $f(z)$ by $\dfrac{f(z+\alpha) - f(\alpha)}{C_N}$.)

Thus, there is $R > 0$ s.t. $f(z) = z^N \left(1 + \sum_{l=1}^{\infty} C_{N+l} z^l\right) \ \forall z \in D(0;R)$.

Let $h(z) = \sum_{l=1}^{\infty} C_{N+l} z^l$. Then for $|z|$ small enough, $|h(z)| < 1$ and we can

define $$g(z) = (1 + h(z))^{1/N} = \sum_{n=0}^{\infty} \frac{\frac{1}{N}\left(\frac{1}{N}-1\right)\cdots\left(\frac{1}{N}-n+1\right)}{n!} h(z)^n$$

i.e. $\exists\ \rho > 0$ s.t. $f(z) = (z \cdot g(z))^N \ \forall z \in D(0;\rho)$. Since $g(0) = 1$, $z \cdot g(z) : D(0;\rho) \to \mathbb{C}$ satisfies the hypothesis of the "inverse function theorem" at 0. Hence we can find $\rho_1, \rho_2 > 0$ such that $z \cdot g(z) : D(0;\rho_1) \xrightarrow{\sim} D(0;\rho_2)$. Hence, f is the composition of $z \mapsto z^N$ with $z \mapsto z \cdot g(z)$

$$D(0;\rho_1^{1/N}) \longrightarrow D(0;\rho_1) \qquad D(0;\rho_1) \xrightarrow{\sim} D(0;\rho_2)$$

and the statement of the proposition follows from the example on page 2.

$\square$

(24.3) Corollary. (Open mapping theorem). Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function, defined on an open and connected set $\Omega \subset \mathbb{C}$. Then $f(\Omega)$ is an open set.

Proof. - Since Ω is connected and f is not constant, by identity theorem (Lecture 22-page6), f is not locally constant. Hence, given $\beta \in f(\Omega)$, say $\beta = f(\alpha)$, $\exists\, N \geq 1$ s.t. $f^{(N)}(\alpha) \neq 0$. By Proposition (24.2) $\exists\, \rho_2 > 0$ s.t. $D(\beta;\rho_2) \subset f(\Omega)$. Hence, $f(\Omega)$ is open. $\square$

(24.4) Isolated singularities. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, where $\Omega \subset \mathbb{C}$ is open and connected. Let $\alpha \in \mathbb{C} \sim \Omega$. We say α is <u>an isolated singularity of f</u> if there exists $R > 0$ such that

$$D^{\times}(\alpha;R) = D(\alpha;R) \sim \{\alpha\} \subset \Omega.$$

i.e. f is defined on <u>all points near α</u> (but perhaps not at α).

Examples: • 0 is an isolated singularity for $\frac{1}{z}$, $\frac{1}{\sin(z)}$, $\frac{1}{e^{z}-1}$.

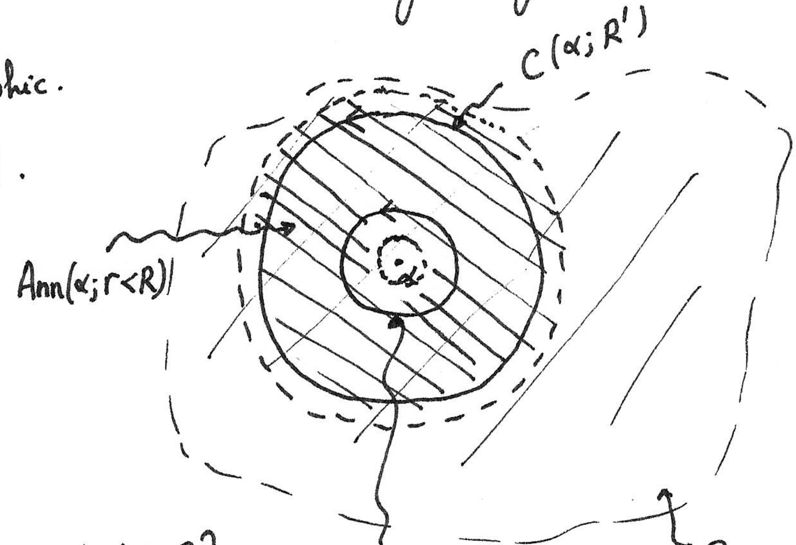
• $z = -1$ is <u>not isolated</u> for $\log(z) : \mathbb{C} \sim \mathbb{R}_{\leq 0} \to \mathbb{C}$.

• 0 is not isolated for $\cancel{\sin(1/z)}$ $\operatorname{cosec}\left(\frac{1}{z}\right)$.

(24.5) Laurent Series expansion near isolated singularity.

Let $f : \Omega \to \mathbb{C}$ be holomorphic.

- α is an isolated singularity of f.
- $R \in \mathbb{R}_{>0}$ is such that $D^{\times}(\alpha; R) \subset \Omega$.



Let $0 \leq r < R$ and define

$$\mathrm{Ann}(\alpha; r < R) = \{ z \in \mathbb{C} : r < |z| < R \}$$

(annular region around α - of inner radius r and outer radius R).

Theorem.* For every $w \in \mathrm{Ann}(\alpha; r < R)$ we have

$$f(w) = \sum_{n=0}^{\infty} c_n (w-\alpha)^n + \sum_{m=1}^{\infty} \frac{d_m}{(w-\alpha)^m} \quad \left(\begin{array}{c} \text{Laurent} \\ \text{Series of} \\ f \text{ near } \alpha \end{array} \right)$$

- $F^{+}(w) = \sum_{n=0}^{\infty} c_n (w-\alpha)^n$ converges uniformly (rel. to cpct sets) on $D(\alpha; R)$
- $F^{-}(w) = \sum_{m=1}^{\infty} d_m (w-\alpha)^{-m}$ converges uniformly (rel. to cpct sets) on $\mathbb{C} - \overline{D}(\alpha; r)$ (i.e. $\{z : |z| > r\}$)

Moreover, (let $r < r' < R' < R$)

$$\boxed{c_n = \frac{1}{2\pi i} \int_{C(\alpha; R')} \frac{f(z)}{(z-\alpha)^{n+1}} \, dz} \; ; \; \boxed{d_m = \frac{1}{2\pi i} \int_{C(\alpha; r')} f(z) \cdot (z-\alpha)^{m-1} \, dz}$$

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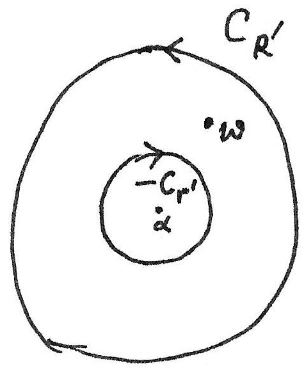
*Pierre Alphonse Laurent (1813-1854) proved this theorem in 1843. The same result was obtained by Weierstrass in 1841 - but Weierstrass' work was not published until 1894 (posthumously).

Proof.- Let $w \in \mathrm{Ann}(\alpha; r < R)$ and choose r', R' $(r < r' < R' < R)$ such that $r' < |w - \alpha| < R'$.



[Note: such r', R' can be chosen for any compact subset of $\mathrm{Ann}(\alpha; r < R)$ - proving uniform convergence (rel. to compact sets) of Laurent Series.]

By Cauchy's integral formula $f(w) = \dfrac{1}{2\pi i} \displaystyle\int_{C_{R'} - C_{r'}} \frac{f(z)}{z - w}\, dz$ (Ex. Why? Hint:

)

$$\Rightarrow f(w) = \frac{1}{2\pi i} \int_{C_{R'}} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \int_{C_{r'}} \frac{f(z)}{z-w}\, dz$$

(i) $F^+(w) = \dfrac{1}{2\pi i} \displaystyle\int_{C_{R'}} \frac{f(z)}{z-w}\, dz$. For $z \in C_{R'}$;

$|z - \alpha| = R' > |w - \alpha|$. Hence

$$\frac{1}{z - w} = \frac{1}{(z-\alpha) - (w - \alpha)} = \frac{1}{z - \alpha} \cdot \frac{1}{1 - \frac{w-\alpha}{z-\alpha}}$$

$$= \sum_{n=0}^{\infty} \frac{(w-\alpha)^n}{(z-\alpha)^{n+1}}$$

(same argument as in Taylor's Theorem See: Lecture 21 page 4)

$$\Rightarrow \quad F^{+}(w) = \frac{1}{2\pi i} \int_{C_{R'}} f(z) \cdot \left(\sum_{n=0}^{\infty} \frac{(w-\alpha)^n}{(z-\alpha)^{n+1}} \right) dz$$

$$= \sum_{n=0}^{\infty} (w-\alpha)^n \cdot \left(\frac{1}{2\pi i} \int_{C_{R'}} \frac{f(z)}{(z-\alpha)^{n+1}} \, dz \right) \qquad \left(\begin{array}{c} \text{using uniform} \\ \text{convergence} \\ \text{of geometric} \\ \text{series} \end{array} \right)$$

(ii) $F^{-}(w) = \dfrac{-1}{2\pi i} \displaystyle\int_{C_{r'}} \frac{f(z)}{z-w} \, dz.$ For $z \in C_{r'}$; $|z-\alpha| = r' < |w-\alpha|$.

Hence,
$$\frac{-1}{z-w} = \frac{1}{(w-\alpha)-(z-\alpha)} = \sum_{m=0}^{\infty} \frac{(z-\alpha)^m}{(w-\alpha)^{m+1}}$$

We get :
$$F^{-}(w) = \sum_{m=1}^{\infty} \frac{1}{(w-\alpha)^m} \cdot \left(\frac{1}{2\pi i} \int_{C_{r'}} f(z) \cdot (z-\alpha)^{m-1} \, dz \right)$$

(Ex. Prove that this series converges uniformly (rel. to compact sets) on $\mathbb{C} \setminus \overline{D}(\alpha; r)$ – using the argument similar to the one we used for Taylor Series – Lecture 21 – page 3).

Combining (i) and (ii) we get

$$f(w) = \underset{\substack{\uparrow \\ \text{holomorphic in} \\ D(\alpha;\, R)}}{F^{+}(w)} + \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{in } \mathbb{C} \setminus \overline{D}(\alpha;\, r)}}{F^{-}(w)} = \sum_{n=0}^{\infty} c_n (w-\alpha)^n + \sum_{m=1}^{\infty} d_m (w-\alpha)^{-m}$$

$\square$