Lecture 25

(25.0) Recall : for $\Omega \subset \mathbb{C}$ (open & connected), $f : \Omega \to \mathbb{C}$ holomorphic ; $\alpha \in \mathbb{C} - \Omega$.

- α is an isolated singularity of f if $\exists\, R > 0$ s.t. $\overset{\times}{D}(\alpha; R) \subset \Omega$

  ($\overset{\times}{D}(\alpha; R) = D(\alpha; R) \setminus \{\alpha\}$ is the punctured disc around α of radius R).

- Laurent Series of f : Choose $r \in [0, R)$.

$$f(z) = f^+(z) + f^-(z) \quad \text{where}$$

(i) $f^+(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n : D(\alpha; R) \to \mathbb{C}$ (uniformly convergent rel. to compact sets)

(ii) $f^-(z) = \sum_{m=1}^{\infty} d_m (z-\alpha)^{-m} : \mathbb{C} - \bar{D}(0, r) \to \mathbb{C}$

Thus, $$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n + \sum_{m=1}^{\infty} \frac{d_m}{(z-\alpha)^m}$$

($z \in \text{Ann}(\alpha; r < R) = \{w \in \mathbb{C} : r < |w - \alpha| < R\}$)

convergence is uniform rel. to compact subsets of $\text{Ann}(\alpha; r < R)$.

(25.1) $f^+(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ is often called the regular part of f.

$f^-(z) = \sum_{m=1}^{\infty} d_m (z-\alpha)^{-m}$: singular part of f.

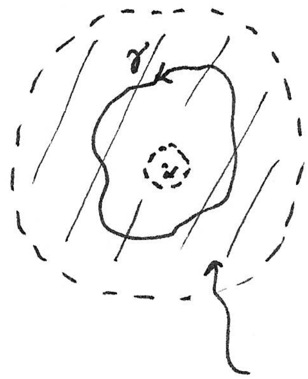
Laurent Series is usually written as a doubly-infinite series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \; ; \quad c_n = \frac{1}{2\pi i} \int_\gamma f(z)(z-\alpha)^{-n-1} dz$$

γ is any contour in $\text{Ann}(\alpha; r < R)$ s.t. $\alpha \in \text{Interior}(\gamma)$.

The coefficient of $(z-\alpha)^{-1}$ in the Laurent Series expansion of f near α is called the residue of f at α.

$$\operatorname{Res}_{\alpha}(f(z)) = c_{-1} = \frac{1}{2\pi i}\int_{\gamma} f(z)\,dz$$



Ann$(\alpha; r<R)$

(25.2) Classification of (isolated) singularities.

With the notations (f, Ω, α) as before,

let $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ be the Laurent Series of f near α.

Definition.- (1) α is called a removable singularity if

$$c_{-1} = c_{-2} = \cdots = 0 \quad \left(\text{i.e. } f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n\right).$$

(2) α is called a pole of order $N \in \mathbb{Z}_{\geq 1}$ if

$$c_{-N} \neq 0 \;;\; c_{-N-1} = c_{-N-2} = \cdots = 0.$$

i.e. $f(z) = \sum_{n=-N}^{\infty} c_n (z-\alpha)^n \quad (c_{-N} \neq 0)$; or $f(z) = \frac{c_{-N}}{(z-\alpha)^N} + \cdots + \frac{c_{-1}}{z-\alpha}$

singular part of α is a polynomial (of degree N) in $\frac{1}{z-\alpha}$.

(3) α is called an essential singularity if

$$\left\{ \underset{(n \in \mathbb{Z}_{\geq 1})}{n} : c_{-n} \neq 0 \right\}$$ is unbounded (or infinite).

(25.3) Typical Examples. - (1) $f(z) = \frac{e^z - 1}{z}$ has removable singularity at $z=0$.

$$\frac{e^z - 1}{z} = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \cdots - 1\right) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

(2) $f(z) = z^{-n}$ $(n \in \mathbb{Z}_{\geq 1})$ has a pole of order n at $z = 0$.

(3) $\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$ has an essential singularity at $z=0$.

(25.4) A pole of order 1 is also called <u>a simple pole</u>.

$z = \infty$ is an isolated singularity (of f) if there exists $R > 0$ such that:

$$\{w \in \mathbb{C} : |w| > R\} \subset \Omega.$$

As a general rule: behavior of $f(z)$ near $z = \infty$ is the same as the behavior of $f(w)$ near $w = 0$; where $w = \frac{1}{z}$.

e.g. $P(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ has a pole of order n at ∞.

e^z, $\sin(z)$, $\cos(z)$ all have an essential singularity at ∞.

(25.5) Alternate characterization - removable singularity

Theorem. $z = \alpha$ is a removable singularity of $f(z)$ if and only if $\lim_{z \to \alpha} (z - \alpha) f(z) = 0$. Equivalently, $f(z)$ is bounded near α; i.e. $\exists\, M \in \mathbb{R}_{>0}$ and $R \in \mathbb{R}_{>0}$ s.t.

$$|f(z)| < M \quad \forall\, z \in D^{\times}(\alpha; R).$$

Proof. We will prove the following implications: $(A) \Rightarrow (C) \Rightarrow (B) \Rightarrow (A)$:

(A) α is removable ; (B) $\lim_{z \to \alpha} (z-\alpha) f(z) = 0$; (C) $\exists\, R>0$ and $M>0$ s.t. $|f(z)| < M$ $\forall\, 0<|z-\alpha|<R$

$(A) \Rightarrow (C)$. If α is a removable singularity then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \text{for } z \in D^{\times}(\alpha; R)$$

$\Rightarrow \lim_{z \to \alpha} f(z) = c_0$. Hence, $\exists\, \rho > 0$ s.t.

$$0 < |z-\alpha| < \rho \Rightarrow |f(z) - c_0| < 1$$

$$\Rightarrow |f(z)| \leq |f(z) - c_0| + |c_0| = |c_0| + 1$$

(i.e. f is bounded on a punctured disc near α).

$(C) \Rightarrow (B)$ by squeeze theorem.

$(B) \Rightarrow (A)$. Define $g(z) = \begin{cases} (z-\alpha)^2 f(z) & ; z \neq \alpha \\ 0 & ; z = \alpha \end{cases}$. $g : \Omega \cup \{\alpha\} \to \mathbb{C}$.

g is holomorphic. $g'(z) = 2(z-\alpha) f(z) + (z-\alpha)^2 f'(z)$; $z \neq \alpha$

$$g'(\alpha) = \lim_{z \to \alpha} \frac{g(z) - g(\alpha)}{z - \alpha} = \lim_{z \to \alpha} (z-\alpha) f(z) = 0.$$

Thus g has a Taylor Series expansion near α (say in $D(\alpha; R)$)

$$g(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^{n+2} \qquad z \in D(\alpha; R)$$

(Note: $g(\alpha) = g'(\alpha) = 0$ as proved above).

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \qquad (z \in D^*(\alpha; R))$$

Hence, by definition, α is a removable singularity of f. □

(25.6) Alternate characterization. Pole of order N.

Theorem. α is a pole of order N $\iff$ $\lim_{z\to\alpha} (z-\alpha)^N f(z)$ exists and is non-zero.

Proof. – ($\Rightarrow$) If f has a pole of order N at $z=\alpha$, then the Laurent Series expansion of f near α has the form

$$f(z) = \frac{c_{-N}}{(z-\alpha)^N} + \cdots + \frac{c_{-1}}{z-\alpha} + \sum_{n=0}^{\infty} c_n (z-\alpha)^n \qquad (c_{-N} \neq 0)$$

$$\Rightarrow \lim_{z\to\alpha} (z-\alpha)^N f(z) = \lim_{z\to\alpha} \sum_{k=-N}^{\infty} c_k (z-\alpha)^{k+N} = c_{-N} \neq 0.$$

($\Leftarrow$) If $\lim_{z\to\alpha} (z-\alpha)^N f(z) = c$ exists and $c \neq 0$, then $g(z) = (z-\alpha)^N f(z)$ satisfies $\lim_{z\to\alpha} (z-\alpha)\, g(z) = 0$. Hence by Theorem (25.5) α is a removable singularity of $g(z)$, i.e.

$$g(z) = c + \sum_{k=1}^{\infty} d_k (z-\alpha)^k \quad \text{near } z=\alpha$$

$$\Rightarrow f(z) = \frac{c}{(z-\alpha)^N} + \cdots$$

near $z=\alpha$; i.e. f has a pole of order N at α □

(25.7) Alternate characterization - Essential Singularity.

Theorem. α is an essential singularity of $f \iff$

$\lim_{z \to \alpha} |f(z)|$ does not exist. (here we are allowing ∞ to be the limit).

Proof. By contradiction, in each direction.

$(\Rightarrow)$ Assume $\lim_{z \to \alpha} |f(z)|$ exists. If it is finite then $f(z)$ is bounded near α and hence α is a removable singularity.

If $\lim_{z\to\alpha} |f(z)| = \infty$, then $\exists\ R>0$ s.t. $|f(z)|>1\ \ \forall z$ s.t. $0<|z-\alpha|<R$

$\Rightarrow h(z) = \dfrac{1}{f(z)}$ is defined on $D^\times(\alpha;R)$ and $|h(z)|<1\ \ \forall z \in D^\times(\alpha;R)$.

By Thm. (25.5) α is a removable singularity of $h(z)$.

$$h(z) = (z-\alpha)^N \left(d_0 + \sum_{k=1}^{\infty} d_k (z-\alpha)^k \right)$$ Taylor Series expansion of $h(z)$ near $z=\alpha$

Here $N =$ order of vanishing of $h(z)$ at $z=\alpha$. ($N \in \mathbb{Z}_{\geq 0}$)

$$\Rightarrow f(z) = \frac{d_0^{-1}}{(z-\alpha)^N} \left(1 + \sum_{k=1}^{\infty} d_k (z-\alpha)^k \right)^{-1}$$

$\Rightarrow \alpha$ is a pole of order N.

($\Leftarrow$) Assume that α is not an essential singularity. Then either α is removable or a pole of order $N \in \mathbb{Z}_{\geq 1}$.

Removable case: If α is removable, $\lim_{z \to \alpha} f(z) = c$ exists $\Rightarrow \lim_{z \to \alpha} |f(z)| = |c|$ also exists.

Pole of order $N \geq 1$: If α is a pole of order N, then

$$\lim_{z \to \alpha} (z-\alpha)^N f(z) = c \qquad (c \neq 0) \text{ exists.}$$

i.e. $\forall\, \varepsilon > 0$, $\exists\, r > 0$ s.t. $0 < |z-\alpha| < r \Rightarrow \left| (z-\alpha)^N f(z) - c \right| < \varepsilon$

i.e. $\left| (z-\alpha)^N f(z) \right| \geq \Big| |c| - |c - (z-\alpha)^N f(z)| \Big| > |c| - \varepsilon$ (for $\varepsilon < |c|$)

e.g. taking $\varepsilon = \frac{|c|}{2}$, we have $r > 0$ s.t.

$$0 < |z-\alpha| < r \Rightarrow \left| (z-\alpha)^N f(z) \right| > \frac{|c|}{2}$$

i.e. $$|f(z)| > \frac{|c|}{2|z-\alpha|^N}$$

Thus given $M \in \mathbb{R}_{>0}$, we can take $\delta < \mathrm{MIN}\left\{ r, \left(\frac{|c|}{2M} \right)^{\frac{1}{N}} \right\}$

So that $$0 < |z-\alpha| < \delta \Rightarrow |f(z)| > M$$

i.e. $\lim_{z \to \alpha} |f(z)| = \infty$ exists $\square$