Lecture 26

(26.0) Recall the notion of an <u>isolated singularity</u> : let $\Omega \subset \mathbb{C}$ be an open and connected set ; $f : \Omega \to \mathbb{C}$ a holomorphic function, $\alpha \in \mathbb{C} \setminus \Omega$.

Then α is an isolated singularity of f if there exists $R > 0$ s.t.

$$D^{\times}(\alpha; R) \subset \Omega.$$

Consider the Laurent series expansion of f near α :

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n .$$

- α is a <u>removable singularity</u> if $c_{-k} = 0 \;\; \forall k \in \mathbb{Z}_{\geq 1}$.

  Equivalently $\lim_{z\to\alpha} (z-\alpha) f(z) = 0$; or $\lim_{z\to\alpha} |f(z)|$ exists and is finite ;

  or, $\exists \rho \in \mathbb{R}_{>0}$ and $M \in \mathbb{R}_{>0}$ s.t. $|f(z)| < M$, $\forall z \in D^{\times}(\alpha;\rho)$.

- α is a <u>pole of order N</u> if $c_{-N} \neq 0$; $c_{-k} = 0 \;\; \forall k \in \mathbb{Z}_{> N}$.

  Equivalently, $\lim_{z\to\alpha} (z-\alpha)^N f(z) = c \neq 0$ exists ;

  or $\lim_{z\to\alpha} |f(z)|$ exists and is infinite.

- α is an <u>essential singularity</u> if $\{ n \in \mathbb{Z}_{\geq 1} : c_{-n} \neq 0 \}$ is infinite.

  Equivalently, $\lim_{z\to\alpha} |f(z)|$ does not exist.

Residue of f at α, denoted by $\operatorname{Res}_{z=\alpha}(f(z))$, is defined as the coefficient of $(z-\alpha)^{-1}$ in the Laurent Series expansion of f near α.

Alternately $\boxed{\operatorname{Res}_{z=\alpha} f(z) = \frac{1}{2\pi i} \int_\gamma f(z)\,dz}$ $\begin{cases} \gamma : \text{contour in } \Omega \text{ s.t.} \\ \alpha \in \text{Interior}(\gamma) \\ \text{Interior}(\gamma) \smallsetminus \{\alpha\} \subset \Omega \end{cases}$

(26.1) Casorati - Weierstrass Theorem. - Assume that α is an essential singularity of f. Then for any $r \in \mathbb{R}_{>0}$, the image under f of the punctured disc $D^\times(\alpha; r)$, is dense in $\mathbb{C}$.

That is, (assuming $D^\times(\alpha;r) \subset \Omega$), $\overline{f(D^\times(\alpha;r))} = \mathbb{C}$.
(closure of a set)

Proof. - By contradiction - let us assume that we have a positive real number $\rho \in \mathbb{R}_{>0}$ so that $D^\times(\alpha;\rho) \subset \Omega$ and $f(D^\times(\alpha;\rho))$ is not dense in $\mathbb{C}$. Meaning, there is a complex number $w \in \mathbb{C}$, and $R \in \mathbb{R}_{>0}$ s.t. $D(w;R) \cap f(D^\times(\alpha;\rho)) = \phi$.

In other words $|f(z) - w| \geq R \quad \forall\ z \in D^\times(\alpha;\rho)$.

Set $h(z) = \dfrac{1}{f(z)-w} : D^\times(\alpha;\rho) \to \mathbb{C}$. Since $h(z)$ is bounded near α; α is a removable singularity of $h(z)$. Hence α is at most a pole of $f(z) = w + \dfrac{1}{h(z)}$, not an essential singularity $\square$

(26.2) Meromorphic* functions. Let $\Omega \subset \mathbb{C}$ be an open, connected set.

A meromorphic function on Ω, often written as $f : \Omega \dashrightarrow \mathbb{C}$ is a holomorphic function on a subset $\Omega' \subset \Omega$ such that every $\alpha \in A = \Omega \setminus \Omega'$ is either a removable singularity of f, or a pole.

That is, f does not have essential singularity, or non-isolated singularity on Ω.

In particular, this definition assumes that $A = \Omega \setminus \Omega'$ consists of isolated points; i.e. $\forall\, \alpha \in A$, $\exists\, r \in \mathbb{R}_{>0}$ s.t. $D(\alpha; r) \cap A = \{\alpha\}$.

Proposition. – With the notations above, if K is a compact set, $K \subset \Omega$; then $A \cap K$ is finite.

Proof. If $A \cap K$ is infinite then by Bolzano-Weierstrass theorem, it must have a cluster point – i.e. $\exists\, \alpha \in A$ s.t. for every $r \in \mathbb{R}_{>0}$ $D(\alpha; r) \cap A$ has infinitely many elements – a contradiction to the statement preceeding this proposition. $\square$

(26.3) Examples. (i) Every holomorphic function $\Omega \to \mathbb{C}$ is meromorphic

(ii) $\operatorname{Cosec}(z) : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic. Let $A = \{n\pi : n \in \mathbb{Z}\} \subset \mathbb{C}$. Then $\operatorname{cosec} : \mathbb{C} \setminus A \to \mathbb{C}$ is holomorphic and has a pole of order 1 at each point of A

* Greek: holo = whole/entire ; Mero = part

(iii) Let $f(z) = \dfrac{P(z)}{Q(z)}$ be a rational function – i.e., P and Q are polynomials. Then $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic.
Its set of singularities $A = \{ z_0 \in \mathbb{C} : Q(z_0) = 0 \}$ is finite.
Assuming P and Q do not have common zeroes; every $z_0 \in A$ is a pole of $f(z)$, of order $=$ order of vanishing of $Q(z)$ at z_0.

(iv) $e^{1/z} : \mathbb{C} \dashrightarrow \mathbb{C}$ is not meromorphic since it has an essential singularity at $z = 0$.

(26.4) Holomorphic functions defined on the entire complex plane are called <u>entire functions</u>. Problem 14 of Set 5 : entire function with a removable singularity at ∞ has to be constant.
Problem 15 of Set 5 : $f : \mathbb{C} \to \mathbb{C}$ an entire function. Assume that ∞ is a pole of order N (of f). ($N \in \mathbb{Z}_{\geq 1}$). Then f is a polynomial of degree N.

Proof. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be the Taylor Series expansion of f near 0. Since f is entire, the radius of convergence of this series is infinite.
Moreover, by assumption, $\displaystyle\lim_{z \to \infty} z^{-N} f(z) = c$ exists and is non-zero.

Meaning : for every $\varepsilon > 0$, we can find $\rho > 0$ s.t.

$$|z| > \rho \quad \Rightarrow \quad |z^{-N} f(z) - c| < \varepsilon$$

$$\text{i.e.} \quad |z^{-N} f(z)| = |z^{-N} f(z) - c + c|$$
$$\leqslant |z^{-N} f(z) - c| + |c| < \varepsilon + |c|.$$

Take, for instance, $\varepsilon = 1$. We get that there exists $R \in \mathbb{R}_{>0}$ s.t.

$$|z| > R \quad \Rightarrow \quad |z^{-N} f(z)| < |c| + 1 =: M.$$

$$\text{i.e.} \quad |f(z)| < M . \cancel{R} |z|^N.$$

We will show that $a_{N+\ell} = 0$ for every $\ell \geq 1$. (Recall: $\sum_{k=0}^{\infty} a_k z^k$ is the Taylor Series expansion of f near $z = 0$). Note that

$$a_{N+\ell} = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{N+\ell+1}} dz$$

C = any contour around 0.
We will take $C = C(0;\rho)$ where $\rho > R$ as found above.

Then
$$|a_{N+\ell}| = \frac{1}{2\pi} \left| \int_{C(0;\rho)} \frac{z^{-N} f(z)}{z^{\ell+1}} dz \right|$$
$$< \frac{1}{2\pi} \frac{M}{\rho^{\ell+1}} \cdot 2\pi\rho = \frac{M}{\rho^{\ell}} \to 0 \text{ as } \rho \to \infty$$

$$\Rightarrow a_{N+\ell} = 0.$$

Hence $f = a_0 + a_1 z + \dots + a_N z^N$ is a polynomial. Note : $a_N = c = \lim_{z \to \infty} z^{-N} f(z) \neq 0 \Rightarrow \text{degree}(f) = N$.

$\square$

(26.5) More generally, assume that $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic.

Prop. - If $z = \infty$ is an isolated singularity of f, then f has finitely many singularities in $\mathbb{C}$. If, in addition, f does not have an essential singularity at ∞ (i.e. ∞ is either removable, or a pole), then f is a rational function.

Proof. Let $A \subset \mathbb{C}$ be the set of singularities of f. Thus $f : \mathbb{C} \setminus A \to \mathbb{C}$ is holomorphic. If ∞ is an isolated singularity of f, then $\exists\, R \in \mathbb{R}_{>0}$ s.t. $\{z \in \mathbb{C} : |z| > R\} \subset \mathbb{C} \setminus A$ (domain of f).

Hence $A \subset \overline{D}(0;R)$. As $\overline{D}(0;R)$ is compact, we obtain that A is finite. (see Prop. 26.2).

Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Consider the Laurent series expansion of f near each α_j $(1 \leq j \leq m)$:

$$f(z) = \sum_{n=1}^{N_j} \frac{d_{j;n}}{(z-\alpha_j)^n} + \sum_{l=0}^{\infty} c_{j;l}\,(z-\alpha_j)^l .$$

Define
$$g(z) = f(z) - \underbrace{\sum_{j=1}^{m} \left(\sum_{n=1}^{N_j} \frac{d_{j;n}}{(z-\alpha_j)^n} \right)}_{\text{rational function of } z \text{, vanishing at } z = \infty.}$$

Claim: $g : \mathbb{C} \to \mathbb{C}$ is holomorphic (i.e. g is entire) and has either a removable singularity, or a pole at ∞.

Given the claim, we conclude that g is a polynomial (using §26.4) and hence $f(z) =$ polynomial + rational ; i.e. a rational function.

Proof of the claim.
$$g(z) = f(z) - \sum_{j=1}^{m} \left(\sum_{n=1}^{N_j} \frac{d_{j;n}}{(z-\alpha_j)^n} \right)$$
is holomorphic on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\}$. Now for each $j = 1, 2, \dots, m$; we have
$$\lim_{z \to \alpha_j} g(z) = \lim_{z \to \alpha_j} \left(f(z) - \sum_{n=1}^{N_j} \frac{d_{j;n}}{(z-\alpha_j)^n} + \sum_{\substack{k \neq j \\ 1 \leq k \leq m}} \sum_{n=1}^{N_k} \frac{d_{k;n}}{(z-\alpha_k)^n} \right)$$
$$= c_{j;0} + \sum_{\substack{k \neq j \\ 1 \leq k \leq m}} \sum_{n=1}^{N_k} \frac{d_{k;n}}{(\alpha_j - \alpha_k)^n}$$
exists and is finite.

$\Rightarrow$ each α_j is a removable singularity of g. Hence $g : \mathbb{C} \to \mathbb{C}$ is an entire function.

$\square$