Lecture 27

(27.0) Residue. For $f : \Omega \to \mathbb{C}$ a holomorphic function and $\alpha \in \mathbb{C} \setminus \Omega$ an isolated singularity of f, we define residue of f at α, denoted by $\operatorname{Res}_{z=\alpha}(f(z))$ as:

$$\boxed{\operatorname{Res}_{z=\alpha}(f(z)) = \frac{1}{2\pi i}\int_{C(\alpha;r)} f(z)\,dz}$$

where

$C(\alpha;r)$ = counterclockwise circle of radius $r>0$ around α. Here $r>0$ is chosen so that $C(\alpha;r) \subset \Omega$ and α is the only singularity of f in $D(\alpha;r)$ (i.e. $D^{\times}(\alpha;r) \subset \Omega$).

Alternately, if $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ is the Laurent Series expansion of f near α, then using $\frac{1}{2\pi i}\int_{\alpha} \frac{1}{(z-\alpha)^n}\,dz = \begin{cases} 1 \,;\, n=-1 \\ 0 \,;\, n \neq -1 \end{cases}$

we get $\operatorname{Res}_{z=\alpha} f(z) = c_{-1}$ = coefficient of $\frac{1}{z-\alpha}$ in the Laurent Series expansion of f near α.

(27.1) Residue at infinity. Assuming ∞ is an isolated singularity of f (i.e. $\exists\, R>0$ s.t. $\{z \in \mathbb{C} \mid |z|>R\} \subset \Omega$)

$$\operatorname{Res}_{\infty}(f(z)) := \frac{1}{2\pi i}\int_{C_R^{-1}} f(z)\,dz$$

C_R^{-1} : clockwise circle of radius R centered at 0. $R>0$ is large enough so that C_R and $\mathbb{C} \setminus \overline{D}(0;R)$ are in Ω.

Note : orientation on C_R^{-1} is "positive from the point of view of ∞" i.e. "∞ remains to the left" while traversing the circle clockwise.

Change of variables $w = z^{-1}$ implies $dz = -w^{-2}\,dw$ and replaces C_R^{-1} by $C_{1/R}$ (counterclockwise circle of radius $\frac{1}{R}$ centered at 0).

We get

$$\operatorname{Res}_{z=\infty}(f(z)) = \frac{1}{2\pi i}\int_{C_{1/R}} -w^{-2} f(w)\,dw$$

$$= -\operatorname{Res}_{w=0}\left(w^{-2} f(w)\right) \qquad \left(w = \frac{1}{z}\right).$$

(27.2) Example .- Let $f(z) = \dfrac{z^3+1}{(z-2)(z-3)}$ singularities at $z=2$, $z=3$ and $z=\infty$.

$$\operatorname{Res}_{z=2} f(z) = \frac{1}{2\pi i}\int_{C_{\cdot 2}} \frac{z^3+1}{(z-2)(z-3)}\,dz = \left[\frac{z^3+1}{z-3}\right]_{\text{set } z=2} = -9.$$

$$\operatorname*{Res}_{z=4}\left(f(z)\right) = \frac{1}{2\pi i}\int_{\bigcirc \cdot 3} \frac{z^3+1}{(z-2)(z-3)}\,dz$$

$$= \left[\frac{z^3+1}{z-2}\right]_{z=3} = 28.$$

$$\operatorname*{Res}_{z=\infty}\left(f(z)\right) = -\operatorname*{Res}_{w=0}\left(\frac{1}{w^2}\,\frac{w^{-3}+1}{(w^{-1}-2)(w^{-1}-3)}\right) = -\operatorname*{Res}_{w=0}\left(\frac{1}{w^3}\,\frac{1+w^3}{(1-2w)(1-3w)}\right)$$

$= -$ Coeff. of w^2 in the Taylor Series expansion of

$$\frac{1+w^3}{(1-2w)(1-3w)} \quad \text{near } w=0.$$

$$\frac{1+w^3}{(1-2w)(1-3w)} = (1+w^3)\left(1+2w+4w^2+\cdots\right)\left(1+3w+9w^2+\cdots\right)$$

Coeff. of $w^2 = 9+4+6 = 19.$

$$\operatorname*{Res}_{z=\infty}\left(f(z)\right) = -19.$$

Note: $\operatorname*{Res}_{z=2}\left(f(z)\right) + \operatorname*{Res}_{z=3} f(z) + \operatorname*{Res}_{z=\infty} f(z)$

$$= -9+28-19 = 0.$$

Can you give an a priori justification why this sum had to be zero?

(27.3) Computing residue.

Assume α is a pole of order N (for f). That is, the Laurent Series expansion of f near α gives :

$$f(z) = \frac{d_N}{(z-\alpha)^N} + \dots + \frac{d_1}{z-\alpha} + \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad (d_N \neq 0)$$

$$= \frac{g(z)}{(z-\alpha)^N} \; ; \text{ where } g(z) = d_N + d_{N-1}(z-\alpha) + \dots + d_1 (z-\alpha)^{N-1} + \sum_{n=0}^{\infty} c_n (z-\alpha)^{n+N}$$

Note : $g(\alpha) = d_N \neq 0$. Hence $\exists\, \rho > 0$ such that $g(z) \neq 0 \;\; \forall z \in D(\alpha; \rho)$.

$$\operatorname*{Res}_{z=\alpha} (f(z)) = \frac{1}{2\pi i} \int_{\alpha} \frac{g(z)}{(z-\alpha)^N} \, dz$$

$$= \frac{g^{(N-1)}(\alpha)}{(N-1)!} \qquad \text{(Cauchy's integral formula)}$$

$= $ Coefficient of $(z-\alpha)^{N-1}$ in the Taylor Series expansion of $g(z)$ near α.

Special case: when $N = 1$ - poles of order $\underline{1}$ are called <u>simple poles</u>.

If α is a simple pole of f, then

$$\operatorname*{Res}_{z=\alpha} f(z) = g(\alpha) \qquad \left(\text{recall: } f(z) = \frac{g(z)}{z-\alpha}\right)$$

$$= \lim_{z \to \alpha} (z-\alpha) f(z).$$

(27.4) More examples.. (i) $\displaystyle \operatorname*{Res}_{z=4} \frac{\sin(z)}{z-4} = \sin(4).$

(ii) $\displaystyle \operatorname*{Res}_{z=0} e^{1/z} = 1 = \text{coefficient of } \frac{1}{z} \text{ in } 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$

(iii) $\displaystyle \operatorname*{Res}_{z=0} \frac{z}{e^z - 1} = 0.$

(iv) $\displaystyle \operatorname*{Res}_{z=n\pi} \operatorname{cosec}(z).$ (Here $n \in \mathbb{Z}$).

Note : $\displaystyle \operatorname*{Res}_{z=\alpha} f(z) = \operatorname*{Res}_{z=0} f(z+\alpha)$

Hence $\displaystyle \operatorname*{Res}_{z=n\pi} \operatorname{cosec}(z) = \operatorname*{Res}_{z=0} \frac{1}{\sin(z+n\pi)}$

$$= (-1)^n \operatorname*{Res}_{z=0} \frac{1}{\sin(z)}. \qquad \left(z=0 \text{ is a simple pole of } \operatorname{cosec}(z) = \frac{1}{\sin(z)}\right)$$

$$= (-1)^n \lim_{z \to 0} \frac{z}{\sin(z)} = (-1)^n.$$

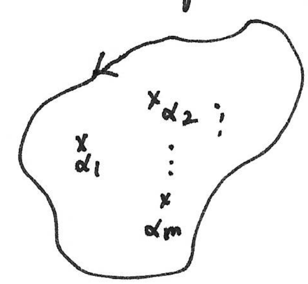
(27.5) <u>Residue Theorem</u>. – Let $f : \Omega \to \mathbb{C}$ be holomorphic.

$\gamma : [a,b] \to \Omega$ a contour in Ω. Let $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be a finite set of points in Interior (γ) so that

- Interior $(\gamma) \sim \{\alpha_1, \dots, \alpha_m\} \subset \Omega$

(i.e, f has isolated singularities at $\alpha_1, \dots, \alpha_m$).

Then
$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{j=1}^{m} \operatorname{Res}_{z=\alpha_j} (f(z))$$



Proof follows from the principle of contour deformation:

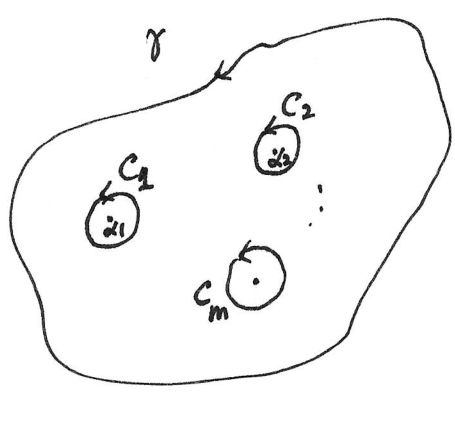
$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{C_j} f(z)\,dz$$

where, for each j, C_j is a small counterclockwise circle around α_j

so that $\begin{cases} C_j \text{ is contained in Interior } (\gamma) \\ \text{for } k \neq j,\ \alpha_k \text{ is outside } C_j \text{ ; } C_j \text{ and } C_k \text{ do not intersect.} \end{cases}$

Hence $\displaystyle \frac{1}{2\pi i} \int_\gamma f(z)\,dz$

$$= \sum_{j=1}^{m} \operatorname{Res}_{z=\alpha_j} (f(z)).$$



$\square$

(27.6) Application to real integrals. - Techniques of contour integration can be used to evaluate real integrals.

I. $\displaystyle\int_0^{2\pi} R(\cos(\theta), \sin(\theta)) \, . \, d\theta .$ Here R is some rational function in 2 variables.

For such integrals, we perform the change of variables : $z = e^{i\theta}$,

so $dz = i \cdot e^{i\theta} \cdot d\theta \implies d\theta = \dfrac{1}{iz} dz$.

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \quad ; \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} .$$

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta)) \, d\theta = \int_{C(0;1)} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz}$$

can be computed using residue theorem

e.g $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}$ $(a > 1)$.

$$= \int_{C(0;1)} \frac{1}{a + \frac{z + z^{-1}}{2}} \cdot \frac{dz}{iz} = \frac{2}{i} \int_{C(0;1)} \frac{dz}{z^2 + 2az + 1} .$$

Roots of $z^2 + 2az + 1 = 0$ are $\alpha = -a + \sqrt{a^2 - 1}$; $\beta = -a - \sqrt{a^2 + 1}$

Note $|\beta| = a + \sqrt{a^2 + 1} > a > 1 \implies \beta$ is not in $\overline{D}(0;1)$

Since $\alpha \cdot \beta = 1$, we get that $\alpha \in D(0;1)$. By residue theorem

$$\frac{2}{i} \int_{C(0;1)} \frac{dz}{(z - \alpha)(z - \beta)} = \frac{2}{i} \cdot 2\pi i \frac{1}{\alpha - \beta} = \frac{2\pi}{\sqrt{a^2 - 1}} .$$

(27.7) Real (infinite) integrals. (Type II): Recall that for a continuous function $Q(x)$, $\int_{-\infty}^{\infty} Q(x)\,dx$ is defined as

$$\int_{-\infty}^{\infty} Q(x)\,dx = \int_{-\infty}^{0} Q(x)\,dx + \int_{0}^{\infty} Q(x)\,dx$$

(i.e. both integrals on the right hand side must individually exist).

$$= \lim_{R_1, R_2 \to \infty} \int_{-R_2}^{R_1} Q(x)\,dx.$$

Cauchy's principal value is defined as:

$$PV \int_{-\infty}^{\infty} Q(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} Q(x)\,dx.$$

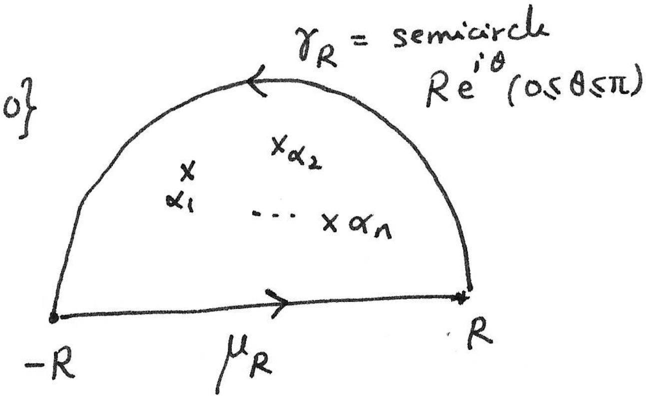
Remarks. - (i) If $\int_{-\infty}^{\infty} Q(x)\,dx$ exists, then it is equal to its Principal value. However, principal value could exist even when $\int_{-\infty}^{\infty} Q(x)\,dx$ does not exist.

e.g. $PV \int_{-\infty}^{\infty} x\,dx = 0$, but $\int_{-\infty}^{\infty} x\,dx$ does not exist.

(ii) In many examples, e.g. if $Q(-x) = Q(x)$ (Q is even), we can check separately that $\int_{-\infty}^{\infty} Q(x)\,dx$ exists. Its principal value can be computed, under certain assumptions, using residue calculations

(27.8) **Theorem.** Let $Q : \Omega \to \mathbb{C}$ be a holomorphic function s.t.

(i) $\exists\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ s.t. $\mathbb{H} \setminus \{\alpha_1, \ldots, \alpha_n\} \subset \Omega$. (also $\mathbb{R} \subset \Omega$).



(ii) $\displaystyle\int_{\gamma_R} Q(z)\, dz \to 0$ as $R \to \infty$.

Then $$PV \int_{-\infty}^{\infty} Q(x)\, dx = \lim_{R \to \infty} \int_{-R}^{R} Q(x)\, dx = 2\pi i \sum_{j=1}^{n} \mathrm{Res}_{\alpha_j}(Q(z))$$

Proof.- By residue theorem ; for $R > \mathrm{Max}\{|\alpha_j| : 1 \le j \le n\}$:

$$\int_{\gamma_R} Q(z)\, dz + \int_{\mu_R} Q(z)\, dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}_{\alpha_j}(Q(z))$$

Take $\lim_{R \to \infty}$ on both sides. $\square$

e.g $$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R \to \infty} \int_{\mu_R} \frac{dz}{z^2+1} = \lim_{R \to \infty} \left(\int_{\gamma_R + \mu_R} \frac{dz}{z^2+1} - \int_{\gamma_R} \frac{dz}{z^2+1} \right)$$

(for even functions $PV \int_{-\infty}^{\infty} = \int_{-\infty}^{\infty}$)

- for z on γ_R ; $|z^2+1| \ge R^2 - 1$ $(R > 1)$

$$\left| \int_{\gamma_R} \frac{dz}{z^2+1} \right| \le \frac{\pi R}{R^2 - 1} \to 0 \text{ as } R \to \infty$$

$$\int_{\gamma_R+\mu_R} \frac{dz}{z^2+1} = \operatorname*{Res}_{z=i}\left(\frac{1}{z^2+1}\right)\cdot 2\pi i$$

$$= \frac{1}{2i}\cdot 2\pi i = \pi .$$

Hence, $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi .$

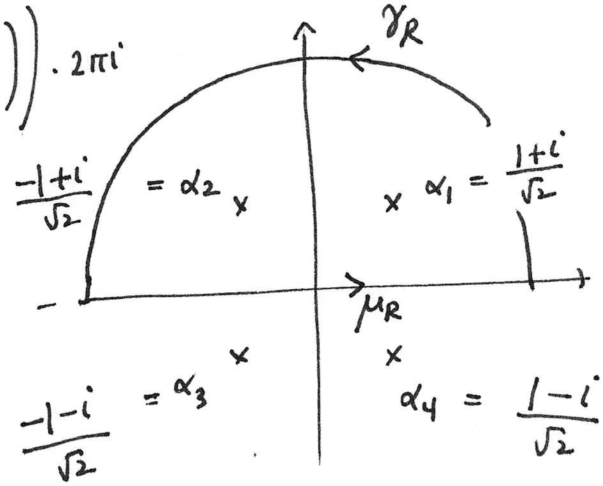
Example. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

$x^4+1=0$ has 4 complex roots

$\alpha_1 = e^{\frac{\pi}{4}i}$ $\quad \alpha_2 = e^{\frac{3\pi}{4}i}$ $\quad \alpha_3 = e^{\frac{5\pi}{4}i}$

$\alpha_4 = e^{\frac{7\pi}{4}i}$

- $\displaystyle\int_{\gamma_R+\mu_R} \frac{dz}{z^4+1} = \left(\operatorname*{Res}_{\alpha_1}\left(\frac{1}{z^4+1}\right) + \operatorname*{Res}_{\alpha_2}\left(\frac{1}{z^4+1}\right)\right)\cdot 2\pi i$



$$= 2\pi i\left(\frac{1}{4\alpha_1^3} + \frac{1}{4\alpha_2^3}\right)$$

(since α_1 is a simple root

$$\operatorname*{Res}_{\alpha_1}\left(\frac{1}{z^4+1}\right) = \lim_{z\to\alpha_1}\frac{z-\alpha_1}{z^4+1} = \lim_{z\to\alpha_1}\frac{1}{4z^3} = \frac{1}{4\alpha_1^3}$$ by l'Hôpital rule.)

$$= 2\pi i\left(-\frac{\alpha_1}{4} - \frac{\alpha_2}{4}\right) = -\frac{\pi i}{2}(\alpha_1+\alpha_2) = \frac{-\pi i}{2}\cdot\sqrt{2}\, i = \frac{\pi}{\sqrt{2}} .$$

- $\displaystyle\lim_{R\to\infty}\int_{\gamma_R}\frac{dz}{z^4+1} = 0$ (same logic as in previous example). Hence, $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}} .$