Lecture 28

(28.0) Recall that we have been studying applications of residue theorem and contour integration techniques towards computing certain types of real integrals.

Residue Theorem: $\boxed{\displaystyle\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}_{\alpha_j}(f(z))}$ where

- $f:\Omega \to \mathbb{C}$ is a holomorphic function defined on an open, connected set Ω.
- $\gamma:[a,b]\to\Omega$ is a contour ; $\alpha_1,\dots,\alpha_m \in \text{Interior}(\gamma)$ s.t.

$$\text{Interior}(\gamma) - \{\alpha_1,\dots,\alpha_m\} \subset \Omega.$$

Real integrals : I. $\displaystyle\int_0^{2\pi} R(\cos(\theta),\sin(\theta))\,d\theta$ is transformed into a contour integral over $C(0;1)$ via change of variables $z=e^{i\theta}$.

II. $$PV\int_{-\infty}^{\infty} Q(x)\,dx := \lim_{R\to\infty}\int_{-R}^{R} Q(x)\,dx = 2\pi i\sum_{j=1}^{N}\operatorname{Res}_{\alpha_j}(Q(z))$$

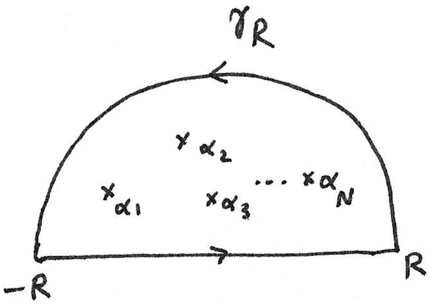
if [assumptions on Q]
- $Q:\Omega\to\mathbb{C}$ is holomorphic. $(\mathbb{R}\subset\Omega)$.
- $\alpha_1,\alpha_2,\dots,\alpha_N \in \mathbb{H}=\{z\in\mathbb{C} : \operatorname{Im}(z)>0\}$ s.t.

$$\mathbb{H}\setminus\{\alpha_1,\dots,\alpha_N\}\subset\Omega$$

- let $\gamma_R(\theta)=Re^{i\theta}$ $(0\le\theta\le\pi)$. Then

$$\lim_{R\to\infty}\int_{\gamma_R} Q(z)\,dz = 0$$

The hypothesis $\int_{\gamma_R} Q(z)\,dz \to 0$ as $R \to \infty$ is satisfied for functions that go to 0 as $z \to \infty$, to order > 1. More precisely, assume $M_R = \text{Max}\{|Q(z)| : z \in \gamma_R\}$. If there exists $\varepsilon > 0$, $C > 0$ such that $M_R < \frac{C}{R^{1+\varepsilon}}$ for R sufficiently large,



then

$$\left| \int_{\gamma_R} Q(z)\,dz \right| < \frac{C}{R^{\varepsilon+\cancel{1}}} \cdot \pi \cdot \cancel{R} \to 0 \quad \text{as } R \to \infty.$$

(28.1) <u>Jordan's Lemma.</u> Let $m \in \mathbb{R}_{>0}$ and let $Q(z)$ be a holomorphic function <u>satisfying the following conditions.</u>

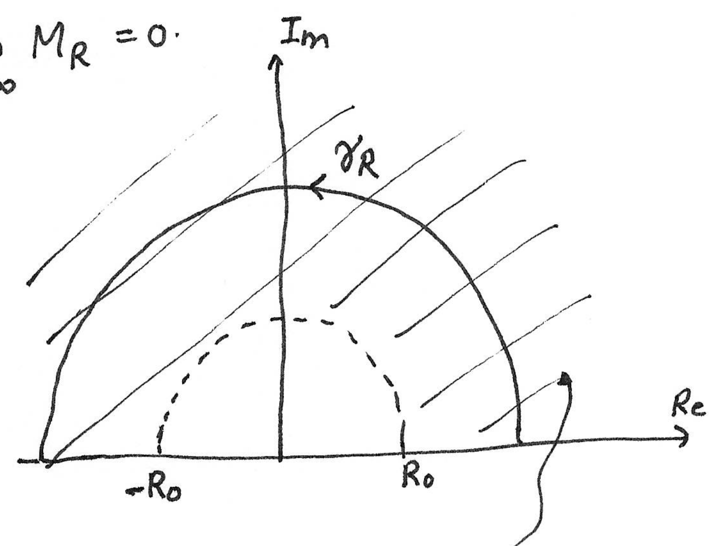
(1) There exists $R_0 \in \mathbb{R}_{>0}$ so that

$$\{z \in \mathbb{C} : \text{Im}(z) \geq 0 \text{ and } |z| > R_0\} \subset \text{Domain of } Q.$$

(2) For every $R > R_0$, let $M_R = \text{Max}\{|Q(z)| : z \in \gamma_R\}$ (where $\gamma_R(\theta) = Re^{i\theta};\ 0 \leq \theta \leq \pi$). Then $\lim_{R \to \infty} M_R = 0$.

Under these assumptions,

$$\lim_{R \to \infty} \int_{\gamma_R} e^{imz} Q(z)\,dz = 0.$$



$\{|z| > R_0 \ \& \ \text{Im}(z) \geq 0\} \subset \text{Domain of } Q.$

Proof. - The proof is based on the following inequality

$$(*) \qquad \sin(\theta) \geq \frac{2\theta}{\pi} \qquad \forall\ \theta \in [0, \tfrac{\pi}{2}].$$

Assuming this, we get

$$\left| \int_\gamma e^{imz} Q(z)\, dz \right| = \left| \int_0^\pi e^{imR(\cos(\theta)+i\sin(\theta))} \cdot Q(Re^{i\theta}) . iRe^{i\theta} d\theta \right|$$

$$< M_R \cdot R . \int_0^\pi e^{-mR\sin(\theta)} d\theta \qquad \left(\begin{array}{l} |Q(z)| \leq M_R \quad \forall z \in \gamma_R . \\ |e^{imR\cos(\theta)}| = 1 \end{array} \right)$$

$$= 2 . M_R . R . \int_0^{\pi/2} e^{-mR\sin(\theta)} d\theta$$

$$\leq 2 M_R R \int_0^{\pi/2} e^{-mR.\frac{2\theta}{\pi}} d\theta = 2 M_R R \left[\frac{-\pi}{2mR} e^{\frac{-2mR\theta}{\pi}} \right]_0^{\pi/2}$$

(using $(*)$)

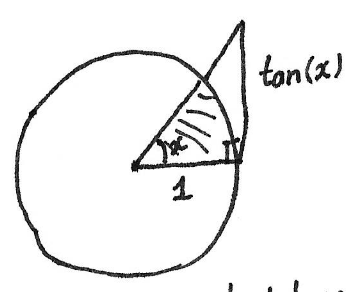
$$= \frac{\pi M_R}{m} \left(1 - e^{-mR}\right)$$

$$\longrightarrow 0 \quad \text{as } R \to \infty \quad \text{since} \quad \lim_{R \to \infty} M_R = 0.$$

Thus the lemma is proved modulo $(*)$.

$\tan(x) > x \quad \forall\ x \in (0, \frac{\pi}{2})$.

Proof of $(*)$. Note that



$\Rightarrow f(x) = \dfrac{\sin(x)}{x}$ is ~~increa~~ a decreasing function

$$f'(x) = \frac{\cos(x)}{x^2}\left(x - \tan(x)\right) < 0 \quad \forall\ x \in (0, \tfrac{\pi}{2}).$$

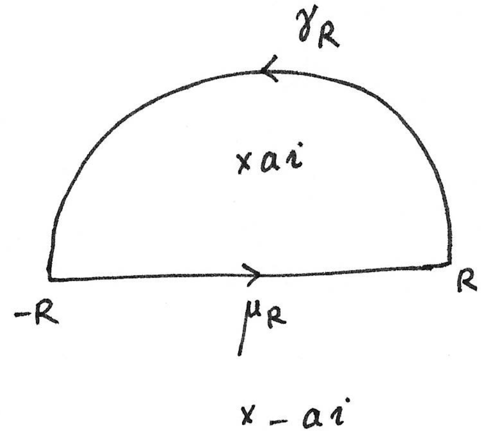
Hence $\dfrac{\sin(x)}{x} \leq \dfrac{\sin(\frac{\pi}{2})}{\pi/2} = \dfrac{2}{\pi}$; $\forall\ x \in (0, \frac{\pi}{2}]$. □

Area of the shaded region $= \frac{1}{2}x < \frac{1}{2}\tan(x) =$ Area of the triangle.

(28.2) Example. Compute $\displaystyle\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx \qquad (a \in \mathbb{R}_{>0})$.

Sol.: Let $Q(z) = \dfrac{z}{z^2+a^2}$ and $m = 1$ in Jordan's Lemma

to conclude that $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} e^{iz} \frac{z}{z^2+a^2}\,dz = 0$



$$\int_{\gamma_R+\mu_R} e^{iz} \cdot \frac{z}{z^2+a^2}\,dz$$

$$= 2\pi i \operatorname*{Res}_{z=ai} \left(e^{iz} \cdot \frac{z}{z^2+a^2} \right)$$

$$= \cancel{2}\pi i \cdot e^{-a} \cdot \frac{\cancel{ai}}{\cancel{2}\cancel{ai}} = \pi i \cdot e^{-a}.$$

Now $\displaystyle\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}\,dx = \operatorname{Im}\left(\frac{1}{2} \lim_{R\to\infty} \int_{\mu_R} \frac{z \cdot e^{iz}}{z^2+a^2}\,dz \right)$

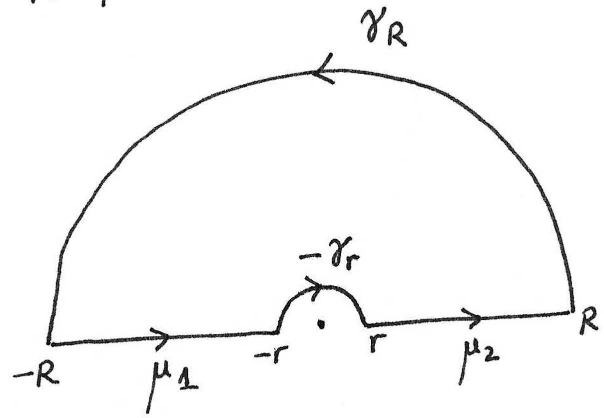
$$= \frac{1}{2} \cdot \operatorname{Im}\left(\lim_{R\to\infty} \int_{\gamma_R+\mu_R} \frac{z \cdot e^{iz}}{z^2+a^2}\,dz \right) \quad \left(\text{since } \lim_{R\to\infty} \int_{\gamma_R} \frac{z e^{iz}}{z^2+a^2}\,dz = 0 \right)$$

$$= \frac{1}{2}\pi \cdot e^{-a}.$$

□

(28.3) Indented contours. – In case the integrand has a singularity on the real axis – say at $x = 0$.

We consider the contour $C_{r,R} = \gamma_R + \mu_1 + \mu_2 - \gamma_r$



- Compute $\displaystyle\int_{C_{r,R}} Q(z)\,dz$ using residues.

- Verify that $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} Q(z)\,dz = 0$

  (using triangle inequality ; or Jordan's lemma)

- Compute $\displaystyle\lim_{r\to 0^+} \int_{\gamma_r} Q(z)\,dz$.

Example (Dirichlet's integral) $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}.$

Consider the contour integral $\displaystyle\int_{C_{r,R}} \frac{e^{iz}}{z}\,dz = 0$ by Cauchy's Theorem

- By Jordan's Lemma, $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0.$

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}}{z}\,dz = \lim_{r\to 0}\int_0^{\pi}\frac{e^{i\cdot r\cdot e^{i\theta}}}{\cancel{r}\cdot \cancel{e^{i\theta}}}\,\cancel{r}\cdot i\cdot \cancel{e^{i\theta}}\,d\theta$$

$$= i\cdot \lim_{r\to 0}\int_0^{\pi} e^{ire^{i\theta}}\,d\theta = i\int_0^{\pi}1\cdot d\theta = \pi i.$$

Hence $$\int_0^{\infty}\frac{\sin(x)}{x}\,dx = \frac{1}{2}\,\mathrm{Im}\left(\lim_{\substack{r\to 0\\ R\to\infty}}\int_{\mu_1+\mu_2}\frac{e^{iz}}{z}\,dz\right)$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\lim_{\substack{r\to 0\\ R\to\infty}}\int_{C_{r,R}}\frac{e^{iz}}{z}\,dz - \int_{\gamma_R}\frac{e^{iz}}{z}\,dz + \int_{\gamma_r}\frac{e^{iz}}{z}\,dz\right)$$

$$= \frac{1}{2}\,\mathrm{Im}(\pi i) = \frac{\pi}{2}.$$

$\square$