Lecture 29

(29.0) Recall: we have been studying real (infinite) integrals using the techniques of contour integration. For instance,

- $$PV \int_{-\infty}^{\infty} Q(x)\,dx = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{\alpha_j}(Q(z))$$; where the

following conditions are assumed: (1) $Q : \Omega \to \mathbb{C}$ holomorphic ($\mathbb{R} \subset \Omega$)

(2) $\alpha_1, \ldots, \alpha_N \in \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$ (upper half plane)

such that $\mathbb{H} \setminus \{\alpha_1, \ldots, \alpha_N\} \subset \Omega$.

(3) for $R \in \mathbb{R}$; $R > |\alpha_j|$, $\forall j = 1, \ldots, N$; let

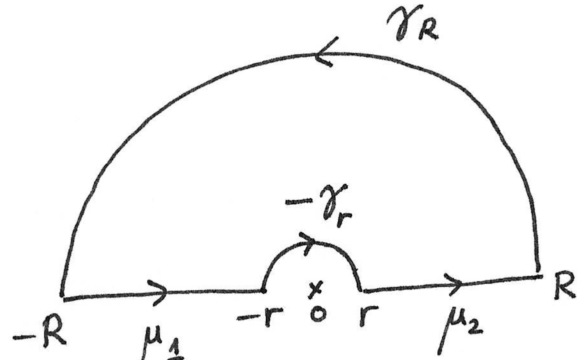
$\gamma_R(\theta) = R e^{i\theta}$ $(0 \leq \theta \leq \pi)$ counterclockwise semicircle of radius R centered at 0; in $\mathbb{H}$.

Then $$\lim_{R \to \infty} \int_{\gamma_R} Q(z)\,dz = 0.$$

Remarks - (a) Jordan's Lemma (28.1) shows (3) for $Q(x) = \cos(mx)\,g(x)$ or $\sin(mx)\,g(x)$, under the hypothesis that $M_R := \operatorname{Max}\{ |g(z)| : z \in \gamma_R \} \to 0$ as $R \to \infty$.

(b) The condition $\mathbb{R} \subset \Omega$ (domain of Q) can be relaxed. to allow $Q(z)$ to have finitely many simple poles on $\mathbb{R}$. In this case we indent our contour to:

(say $Q(z)$ has a pole at 0).



$$\text{P.V.} \int_{-\infty}^{\infty} Q(x)\,dx := \lim_{\substack{r \to 0 \\ R \to \infty}} \int_{\mu_1 + \mu_2} Q(x)\,dx$$

$$C_{r,R} = \gamma_R + \mu_1 - \gamma_r + \mu_2 .$$

$$= \lim_{\substack{r \to 0 \\ R \to \infty}} \left(\int_{C_{r,R}} Q(z)\,dz - \int_{\gamma_R} Q(z)\,dz + \int_{\gamma_r} Q(z)\,dz \right)$$

$$= 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{\alpha_j} Q(z) - \underbrace{\lim_{R \to \infty} \int_{\gamma_R} Q(z)\,dz}_{\substack{\text{(0 in most} \\ \text{examples)}}} + \underbrace{\lim_{r \to 0^+} \int_{\gamma_r} Q(z)\,dz}_{\substack{\text{can be computed easily.} \\ \text{(see Lemma below)}}}$$

($\alpha_1, \ldots, \alpha_N$ = singularities (isolated) of Q in $\mathbb{H}$)

(29.1) Lemma. Assume $Q(z)$ is holomorphic and $z = 0$ is a simple pole of Q. Then,

$$\lim_{r \to 0} \int_{\gamma_r} Q(z)\,dz = \pi i \operatorname{Res}_{z=0} (Q(z)) .$$

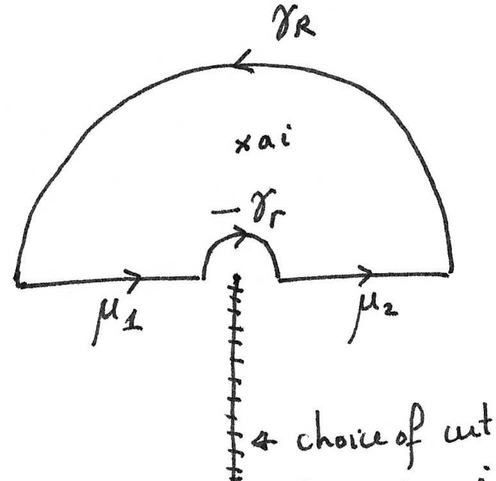
Proof. Since $z=0$ is a simple pole of Q, in a disc around 0, we can write $Q(z) = \frac{\psi(z)}{z}$ $(\psi(0) \neq 0)$.

$$\Rightarrow \int_{\gamma_r} Q(z)\,dz = \int_0^{\pi} \frac{\psi(r\cdot e^{i\theta})}{\cancel{r}\,\cancel{e^{i\theta}}}\,\cancel{r}\cdot i\cdot \cancel{e^{i\theta}}\,d\theta$$

$$= i \int_0^{\pi} \psi(r\cdot e^{i\theta})\,d\theta. \qquad \psi(0) = \operatorname{Res}_0 Q(z)$$

$$\xrightarrow[\text{as } r\to 0]{} i\cdot \operatorname{Res}_0 Q(z)\cdot\pi = \pi i \operatorname{Res}_0 Q(z) \qquad \square$$

(29.2) Example $\displaystyle\int_0^{\infty} \frac{\ln(x)}{x^2+a^2}\,dx.$



- Choice of "$\log(z)$".

$\mathcal{C} = \mathbb{R}_{\leq 0}\, i$ (choice of a cut)

$\log_{\mathcal{C}} : \mathbb{C}\setminus\mathcal{C} \to \mathbb{C}$ defined by

$$\log_{\mathcal{C}}(z) = \ln|z| + i\,\theta$$

$$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}\,; \quad \theta = \arg(z) \pmod{2\pi}.$$

- $$\int_{C_{r,R}} \frac{\log_{\mathcal{C}}(z)}{z^2+a^2}\,dz = 2\pi i \operatorname{Res}_{ai}\left(\frac{\log_{\mathcal{C}}(z)}{(z-ai)(z+ai)}\right)$$

$$= 2\pi i\left(\frac{\log_{\mathcal{C}}(ai)}{2ai}\right) = \frac{\pi}{a}\left(\ln(a) + \frac{\pi}{2} i\right)$$

- $$\left| \int_{\gamma_R} \frac{\log_e(z)}{z^2+a^2}\, dz \right| \leqslant \frac{\sqrt{2}\, \ln(R)}{R^2-a^2} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

$$\left(\begin{array}{l} z = Re^{i\theta} \\ \theta \in [0,\pi] \end{array} ; \quad |\log_e(z)| = \sqrt{\ln(R)^2 + \theta^2} \leq \sqrt{2}\cdot \ln(R) \text{ if } \ln(R) > \theta \right)$$

- $$\left| \int_{\gamma_r} \frac{\log_e(z)}{z^2+a^2}\, dz \right| \leqslant \frac{\sqrt{2}\, |\ln(r)|}{a^2-r^2} \cdot \pi r \to 0 \text{ as } r \to 0.$$

for small $r \in (0,1)$

$$\Rightarrow \lim_{\substack{r\to 0 \\ R\to\infty}} \int_{C_{r,R}} \frac{\log_e(z)}{z^2+a^2}\, dz = \lim_{\substack{r\to 0 \\ R\to\infty}} \int_{-R}^{-r} \frac{\log_e(z)}{z^2+a^2}\, dz + \int_r^R \frac{\log_e(z)}{z^2+a^2}\, dz$$

Now $$\int_{-R}^{-r} \frac{\log_e(z)}{z^2+a^2}\, dz = -\int_R^r \frac{\log_e(-x)}{x^2+a^2}\, dx \qquad (x = -z)$$

$$= \int_r^R \frac{\ln(x) + \pi i}{x^2+a^2}\, dx$$

$$\Rightarrow \frac{\pi}{a}\left(\ln(a) + \frac{\pi}{2} i \right) = 2\int_0^\infty \frac{\ln(x)}{x^2+a^2}\, dx + \pi i \int_0^\infty \frac{dx}{x^2+a^2}.$$

Real part : $$\int_0^\infty \frac{\ln(x)}{x^2+a^2}\, dx = \frac{\pi}{2a} \ln(a) ;$$ Imaginary part $$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}.$$

(29.3) Application of residues continued: counting zeroes and poles of meromorphic functions using logarithmic derivative

Let $f : \Omega \dashrightarrow \mathbb{C}$ be a meromorphic function (i.e. there exists $A \subset \Omega$ s.t. $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic and every $\alpha \in A$ is either a removable singularity of f, or a pole of f).

Given $\alpha \in A$ (assume α is a pole), Laurent series expansion of f near α gives

$$f(z) = \frac{\psi(z)}{(z-\alpha)^{N_\alpha}}$$

($N_\alpha \in \mathbb{Z}_{\geq 1}$ order of pole at α)

$\psi(z) \neq 0 \quad \forall z \in D(\alpha;\rho)$ (for some $\rho \in \mathbb{R}_{>0}$)

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{-N_\alpha}{z-\alpha} + \frac{\psi'(z)}{\psi(z)}$$ ← holomorphic near α

$$\Rightarrow \quad \operatorname*{Res}_{z=\alpha}\left(\frac{f'}{f}\right) = -\text{ order of the pole at } \alpha.$$

Similarly, if f has a zero of order N at $z_0 \in \Omega$.

$$f(z) = (z-z_0)^N \cdot \varphi(z)$$ ($\varphi(z) \neq 0 \quad \forall z \in D(z_0;\rho)$)

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{N}{z-z_0} + \frac{\varphi'(z)}{\varphi(z)}$$ ← holomorphic near z_0.

Assume (without losing any generality) that every $\alpha \in A$ is a pole of f. The previous calculation shows that

Proposition.— (1) $\dfrac{f'}{f} : \Omega \dashrightarrow \mathbb{C}$ has only simple poles at points from A; and zeroes of f $\left(Z(f) = \{ z \in \Omega \setminus A : f(z) = 0 \} \right)$.

(2) Let γ be a contour in Ω so that

$$\gamma \cap \left(A \cup Z(f) \right) = \phi$$

(f does not have zeroes or poles on γ). Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \begin{array}{c}\text{Number of zeroes}\\ \text{of } f \\ \text{within } \gamma\end{array} - \begin{array}{c}\text{Number of poles}\\ \text{of } f \\ \text{within } \gamma\end{array}$$

[counted with multiplicity]

$$= \sum_{z_0 \in Z(f) \cap \text{Interior}(\gamma)} N_{z_0} - \sum_{\alpha \in A \cap \text{Interior}(\gamma)} N_\alpha .$$

N_z = order of vanishing / order of pole

(29.4) <u>Logarithm of a function.</u> Let $f : \Omega \to \mathbb{C}$ be holomorphic, where $\Omega \subset \mathbb{C}$ is (in addition to being open, connected) simply-connected. Assume that $f(z) \neq 0$ $\forall z \in \Omega$. Then $\exists\, g : \Omega \to \mathbb{C}$ holomorphic such that $e^{g(z)} = f(z)$ $\forall z \in \Omega$.

Proof. - Since $f(z) \neq 0$ $(\forall z \in \Omega)$; $\frac{f'(z)}{f(z)} : \Omega \to \mathbb{C}$ is again holomorphic. As Ω is simply-connected, every holomorphic function on Ω has an antiderivative. Namely, choose $z_0 \in \Omega$ and define

$$g(z) = \int_\gamma \frac{f'(z)}{f(z)}\, dz \qquad \left(\text{antiderivative of } \frac{f'}{f}\right)$$

(joining z_0 to z)

This definition is independent of chosen path γ, since $\frac{f'}{f}$ is holomorphic and Ω is simply-connected.

Claim: $f(z_0)\cdot \exp(g(z)) = f(z)$ $(\forall z \in \Omega)$.

Let $h(z) = \dfrac{f(z)\cdot e^{-g(z)}}{f(z_0)}$. $\quad \left(h(z_0) = \dfrac{f(z_0)\cdot e^{-g(z_0)}}{f(z_0)} = 1 \right)$

(Note: $g(z_0) = 0$)

$$h'(z) = \frac{1}{f(z_0)} \left(f'(z)\cdot e^{-g(z)} - f(z)\cdot e^{-g(z)} g'(z) \right)$$

$$= \frac{1}{f(z_0)}\, e^{-g(z)} \left(f'(z) - \cancel{f(z)}\cdot \frac{f'(z)}{\cancel{f(z)}} \right) = 0.$$

Thus $h(z) = 1$ $(\forall z \in \Omega)$ $\Rightarrow$ $f(z) = \exp\big(\underbrace{\log(f(z_0))}_{\text{any choice of log.}} + g(z)\big)$ $\square$