Lecture 30

(30.0) Recall that last time we studied the logarithmic derivative $\dfrac{f'(z)}{f(z)}$ for a meromorphic function $f : \Omega \dashrightarrow \mathbb{C}$. We proved (see Prop. 29.3, page 6 of Lecture 29)

(1) $\dfrac{f'}{f} : \Omega \dashrightarrow \mathbb{C}$ has <u>only simple poles</u> (i.e. poles of order 1) at $\mathcal{Z} = \{z \in \Omega : f(z) = 0\}$ and $\mathcal{P} \subset \Omega$ (set of poles of f)

(i.e. $f : \Omega \setminus \mathcal{P} \to \mathbb{C}$ is holomorphic; and every $\alpha \in \mathcal{P}$ is a pole of f)

(2) For $z_0 \in \mathcal{Z}$, $\operatorname{Res}_{z_0}\left(\dfrac{f'}{f}\right) =$ order of vanishing of f at z_0.

For $\alpha \in \mathcal{P}$, $\operatorname{Res}_{\alpha}\left(\dfrac{f'}{f}\right) = -$ order of the pole of f at α.

(3) Let $\gamma : [0,1] \to \Omega \setminus \{\mathcal{P} \cup \mathcal{Z})$ be a contour. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{Number of zeroes of } f \text{ within } \gamma - \text{Number of poles of } f \text{ within } \gamma$$

[each counted with multiplicity].

In particular,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz \in 2\pi i\,\mathbb{Z}.$$

We also showed that if $F : \Omega \to \mathbb{C}$ is nowhere vanishing, holomorphic function, then $F(z) = \exp(G(z))$ for a holomorphic $G : \Omega \to \mathbb{C}$.

(30.1) Rouché's Theorem. ($\Omega \subset \mathbb{C}$ open connected set)

Let $f, g : \Omega \dashrightarrow \mathbb{C}$ be two meromorphic function. Let

$Z(f) = \{z \in \Omega \mid f(z) = 0\}$ Similarly $Z(g)$ (Zeroes of f & g resp.)

$P(f)$ = set of poles of $f \subset \Omega$
(similarly $P(g)$).

Let γ be a contour in $\Omega \setminus (Z(f) \cup Z(g) \cup P(f) \cup P(g)) =: \Omega'$

$Z_\gamma(f) = |Z(f) \cap \text{Interior}(\gamma)|$ counted with multiplicity – similarly $P_\gamma(f)$, $Z_\gamma(g)$, $P_\gamma(g)$.

Assume that $\forall z \in \gamma$; $|f(z) + g(z)| < |f(z)| + |g(z)|$.

Then, $Z_\gamma(f) - P_\gamma(\overset{f}{\cancel{g}}) = Z_\gamma(g) - P_\gamma(g)$.

Proof.– By Prop. 29.3,

$$\frac{1}{2\pi i} \int_\gamma \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right) dz = Z_\gamma(f) - P_\gamma(f) - Z_\gamma(g) + P_\gamma(g)$$

L.H.S. is the integral of the logarithmic derivative of $\delta(z) = \dfrac{f(z)}{g(z)}$. Note that,

for $z \in \gamma$; $\delta(z) \notin \mathbb{R}_{\geq 0}$ – since in that case $|\delta(z) + 1| = |\delta(z)| + 1$
i.e. $|f(z) + g(z)| = |f(z)| + |g(z)|$
contradicting the strict inequality.

By continuity of $\delta : \Omega' \to \mathbb{C}$, we can choose an open set $\mathcal{U} \subset \Omega'$; $\gamma \subset \mathcal{U}$ so that $\delta(\mathcal{U}) \subset \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

i.e. $\delta : \mathcal{U} \to \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Composing with a branch of log – say

$\log_+ : \mathbb{C} \setminus \mathbb{R}_{\geq 0} \to \mathbb{C}$ $\left(\log_-(z) = \ln|z| + i\theta \right.$
$\left. 0 < \theta < 2\pi \,;\, \theta = \arg(z) \pmod{2\pi} \right)$

we have $\log \delta : \mathcal{U} \to \mathbb{C}$ so that $\frac{d}{dz}(\log \delta) = \frac{\delta'}{\delta}$
(primitive of $\frac{\delta'}{\delta}$).

$\Rightarrow \quad \frac{1}{2\pi i} \int_\gamma \frac{\delta'(z)}{\delta(z)}\, dz = 0$. This proves Rouché's theorem

□

(30.2) Remarks: (1) Assuming f and g are holomorphic on Ω

and $\frac{f}{g} = 1 + \phi(z)$, so that $f + g = g(1+\phi(z))$ –

Rouché's Theorem can be used to show that :

if $|\phi(z)| < 1$ for every $z \in \gamma$, then g and $g(1+\phi(z))$

has the same number of zeroes within γ (reformulation of the theorem).

(Proof. $\quad Z_\gamma(g(1+\phi(z))) - Z_\gamma(g) = \frac{1}{2\pi i} \int_\gamma \frac{\phi'(z)}{1+\phi(z)}\, dz$

On γ; $|\phi(z)| < 1 \Rightarrow \phi(z) + 1 \in D(1;1)$. Composing

with $\log : D(1;1) \to \mathbb{C}$, we conclude that $\frac{1}{2\pi i} \int_\gamma \frac{\phi'(z)}{1+\phi(z)}\, dz = 0$.)

(2) Alternate proof of the fundamental theorem of algebra can

be given using Rouché's Thm. Let $P(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$ be

a polynomial of degree n. Then

$$P(z) = z^n \left(1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right) = z^n(1+\phi(z)).$$

For $z \in \gamma = C(0;R)$ (circle $|z| = R$) ; $\quad |\phi(z)| \leq \frac{|a_{n-1}|}{R} + \dots + \frac{|a_0|}{R^n} < 1$ for R sufficiently large.

$\Rightarrow$ Within $C(0;R)$, z^n and $P(z)$ have the same number of zeroes ; i.e. n.

(3) In concrete examples, Rouché's Thm. can be used to estimate number of solutions of an equation $f(z) = A$ ($A \in \mathbb{C}$ fixed).

e.g. let $P(z) = z^4 + 8z + 1$. On $\gamma = C(0;1)$ (i.e. $|z| = 1$)

we have $8 = 8|z| > |z^4 + 1|$ (since $|z^4+1| \leq |z|^4 + 1 = 2$.)

$\Rightarrow$ $P(z)$ and $8z$ have the same number of zeroes within $D(0;1)$ i.e. 1.

$\Rightarrow$ $\exists!$ solution of $z^4 + 8z + 1 = 0$ in $|z| < 1$.

(30.3) Integral representation of the (local) inverse of a holomorphic function.

Recall (Lecture 23 - §23.1 - every power series $A(z)$ with non-zero coefficient of z^1, has a local inverse ; - also Lecture 24, §24.1).

if $f : \Omega \to \mathbb{C}$ is a holomorphic function; $\alpha \in \Omega$ is s.t. $f'(\alpha) \neq 0$, then $\exists\ \rho_1, \rho_2 > 0$ s.t. $f : D(\alpha; \rho_1) \to D(f(\alpha), \rho_2)$ has a holomorphic inverse

Note:
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - A}\, dz = \text{Number of solutions of } f(z) = A.$$
$$= 1 \quad \text{if } A \in D(f(\alpha); \rho_2).$$

($\gamma = C(\alpha; \rho_1)$). And $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z) - A}\, dz = \alpha$ so that $f(\alpha) = A$

i.e. $$f^{-1}(A) = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z)-A}\, dz$$ (right-hand side viewed as a function of A)

(30.4) Winding number of a closed curve. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed (piecewise smooth, not necessarily simple) curve.

Let $C_\gamma = \{\gamma(t) : 0 \le t \le 1\} \subset \mathbb{C}$ (image of γ in $\mathbb{C}$).

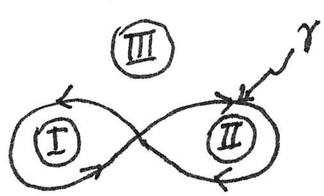
For $z_0 \in \mathbb{C} \setminus C_\gamma$, define $$\boxed{W_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}}$$ winding number of γ around z_0.

Since $\frac{1}{z-z_0}$ = logarithmic derivative of $z - z_0$, $W_\gamma(z_0) \in \mathbb{Z}$ for every $z_0 \in \mathbb{C} \setminus C_\gamma$.

Remarks. (1) $W_\gamma(z_0)$ is sometimes also called the index of γ around z_0, and some textbooks denote it by $\mathrm{Ind}_\gamma(z_0)$.

(2) $W_\gamma(z_0)$ counts how many times γ circles around z_0, counted with a sign of $+1$ for counterclockwise, -1 for clockwise circling.

e.g.



$$W_\gamma(z_0) = \begin{cases} +1 \,; & z_0 \in \text{I} \\ -1 \,; & z_0 \in \text{II} \\ 0 \,; & z_0 \in \text{III} \end{cases}$$