Lecture 31

(31.0) In this lecture, we are going to discuss a remarkable application of the results we have learnt so far:

- Cauchy's integral formula and principle of contour deformation (Lecture 13)
- Weierstrass' Theorem on uniformly convergent seq. of functions (Lecture 19)

and the very important inequality

- $\displaystyle \left| \int_\gamma f(z)\,dz \right| \leq \mathrm{Max}\{|f(z)| : z \in \gamma\} \cdot \text{Length of } \gamma$ (Lecture 11).

We will use these to obtain a few identities involving trigonometric functions (almost all due to Euler*). For instance,

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$\cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

$$\mathrm{Cosec}(z) - \frac{1}{z} = \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2\pi^2}$$

One can view these identities as approximating trigonometric functions by rational functions - the last two are often viewed as "partial fraction decomposition" - except now there are infinitely many terms.

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* Leonhard Euler 15/4/1707 - 18/9/1783

(31.1) Euler and the Basel problem. [Optional]

The Basel problem was to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$. It was first posed by Pietro Mengoli (1625–1686) in 1650, and later popularized by Jacob Bernoulli (6/1/1655 - 16/8/1705, Basel, Switzerland).

This problem was solved by Euler in 1734, using the formula (due to John Wallis 23/11/1616 - 28/10/1703):

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} 1 - \frac{z^2}{n^2\pi^2}$$

- $\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$ (Taylor Series)
- Coefficient of z^2 in the product $\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\cdots$

$$= -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Comparing the two answers we get $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$

In principle, we can get a method of computing $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ $\forall\, k \geq 1$ from this. e.g. Coeff. of z^4 in the Taylor series of $\frac{\sin(z)}{z}$ $= \frac{1}{5!} = \frac{1}{120}$

Coefficient of z^4 in the infinite product $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$

$$= \frac{1}{\pi^4}\sum_{\substack{m_1<m_2 \\ (m_1,m_2\in\mathbb{Z}_{\geq 1})}} \frac{1}{m_1^2 m_2^2}.$$

Now, $$2\sum_{m_1<m_2}\frac{1}{m_1^2m_2^2} = \left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^2 - \sum_{m=1}^{\infty}\frac{1}{m^4}$$

$$\Longrightarrow \quad \frac{1}{\pi^4}\cdot\frac{1}{2}\left(\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^2 - \sum_{m=1}^{\infty}\frac{1}{m^4}\right) = \frac{1}{5!} = \frac{1}{120}.$$

Setting $\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$, we get $\frac{\pi^4}{36} - \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{60}$

$$\Longrightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4\left(\frac{1}{36}-\frac{1}{60}\right) = \frac{\pi^4}{90}.$$

In order to carry such computations out for any $k\in\mathbb{Z}_{\geq 1}$ uniformly, Euler considered the logarithmic derivative of $\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$:

$$\cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2\pi^2}$$

$$= -2z\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-z^2} = -2z\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(1-\frac{z^2}{n^2\pi^2}\right)^{-1}$$

$$= -2z\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(\sum_{\ell=0}^{\infty}\frac{z^{2\ell}}{n^{2\ell}\pi^{2\ell}}\right) \quad \text{(geometric series)}$$

$$\cot(z) - \frac{1}{z} = -2z \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \left(\sum_{l=0}^{\infty} \frac{z^{2l}}{n^{2l}\pi^{2l}} \right)$$

$$= -2z \sum_{N=0}^{\infty} z^{2N} \left(\sum_{n=1}^{\infty} \frac{1}{(n\pi)^{2(N+1)}} \right)$$

i.e. $$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \pi^{2k} \cdot \left(\frac{\text{Coefficient of } z^{2k-1} \text{ in } \cot(z) - \frac{1}{z}}{-2} \right)$$

$$= -\frac{\pi^{2k}}{2} \cdot \left(\text{coefficient of } z^{2k-1} \text{ in the Taylor Series of } \cot(z) - \frac{1}{z} \right)$$

Taylor Series of $\cot(z) - \frac{1}{z}$ (in term of Bernoulli numbers - Lecture 21, §21.3 - Problem 14, 15 of Set 4)

$$\cot(z) - \frac{1}{z} = i \left(\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right) - \frac{1}{z}$$

$$= i \left(\frac{e^{iz} - e^{-iz} + 2 \cdot e^{-iz}}{e^{iz} - e^{-iz}} \right) - \frac{1}{z}$$

$$= i \left(1 + \frac{2}{e^{2iz} - 1} \right) - \frac{1}{z} = i \left(1 + \frac{1}{iz} \frac{2iz}{e^{2iz} - 1} \right) - \frac{1}{z}$$

$$= i \left(1 + \frac{1}{iz} \left(1 - \frac{1}{2}(2iz) + \sum_{k=1}^{\infty} B_{2k} \frac{(z \cdot 2i)^{2k}}{(2k)!} \right) \right) - \frac{1}{z}$$

$$= i + \frac{1}{z} \left(1 - iz + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2^{2k} \cdot B_{2k}}{(2k)!} z^{2k} \right) - \frac{1}{z}$$

$$\boxed{\text{recall: } \frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!} \; ; \; B_{2k}\text{'s - Bernoulli numbers}}$$

$$\Rightarrow \quad \cot(z) - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2^{2k} \cdot B_{2k}}{(2k)!} z^{2k-1} \qquad \left(\text{radius of convergence} = 2\pi\right)$$

Hence

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} \cdot 2^{2k-1} \cdot B_{2k}}{(2k)!} \cdot \pi^{2k}}$$

Using the recurrence relation $\sum_{l=0}^{n} \binom{n+1}{l} B_l = 0$; $B_0 = 1$, $B_1 = \frac{-1}{2}$
(Problem 14, Set 4) $B_{2l+1} = 0 \quad \forall\, l \geq 1$

one can compute first few Bernoulli numbers:

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{-1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{-1}{30},$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad \ldots$$

(31.2) For further reading about such fascinating computations, see

V. S. Vardharajan : Euler and his work on infinite series
Bulletin of the American Mathematical Society (Oct. 2007)

For instance, Euler's earlier attempts to solve the Basel problem led him to define "multilogarithms $Li_k(z)$" and zeta function $\zeta(s)$

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \qquad \left(\text{radius of convergence} = 1\right)$$
$$(k \in \mathbb{Z}_{\geq 1})$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\operatorname{Re}(s) > 1)$$

or combining these $\quad \mathcal{L}(z;s) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^s}$

$$\begin{tikzcd}
 & \mathcal{L}(z;s) \arrow[dl, "z \to 1"'] \arrow[dr, "s = k \in \mathbb{Z}_{\geq 1}"] & \\
\zeta(s) \arrow[dr, "s = k"', "(k \geq 2)"'] & & \mathrm{Li}_k(z) \arrow[dl, "\lim_{z \to 1}"] \\
 & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k} &
\end{tikzcd}$$

("Abel's Theorem")