Lecture 32

(32.0) Generalized partial fractions.

Certain type of meromorphic functions can be written as an infinite sum of rational functions (or simple fractions $\frac{1}{z-b}$) - analogous to the partial fraction decomposition of rational functions. This result is due to Gösta Mittag Leffler (1846-1927) in 1880 ; and is an instance of a more general theorem (also due to Mittag-Leffler).

(32.1) Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function. Let us assume that 0 is not a pole of f. (This is not a serious assumption since if f did have a pole at 0, we can replace f by f − Singular part of f near 0).

Let $A \subset \mathbb{C}$ be the set of poles of f. Recall that since f is a meromorphic function, A has to be a discrete set, i.e. $\forall R \in \mathbb{R}_{>0}$

$$|A \cap \overline{D}(0;R)| < \infty.$$

Thus we can arrange $A = \{a_1, a_2, \ldots\}$ so that

$$0 < |a_1| \leq |a_2| \leq \cdots$$

Now we impose two conditions on f:

<u>Assumption 1</u>. - All poles of f are simple (i.e., of order 1).
Let $b_k = \mathrm{Res}_{a_k}(f)$ $(\forall\ k \geq 1)$.

<u>Assumption 2.</u> For every $m \in \mathbb{Z}_{\geq 1}$, it is possible to choose $R_m \in \mathbb{R}_{>0}$ in such a manner that

(1) $R_1 < R_2 < \cdots$ and $\lim\limits_{m \to \infty} R_m = \infty$.

(2) Let $C_m = C(0; R_m)$ circle of radius R_m centered at 0.
Then $A \cap C_m = \phi$ (i.e. f has no poles on C_m).

(3) $\exists\, M \in \mathbb{R}_{>0}$ s.t. $|f(z)| < M \quad \forall\, z \in C_m \quad \forall\, m \geq 1$.

Theorem : [f, A as above, satisfying Assumptions 1 & 2]

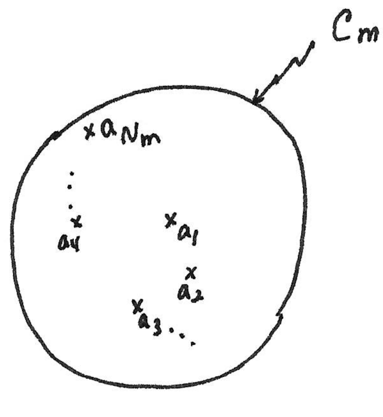
$$\boxed{f(w) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{w - a_k} + \frac{1}{a_k} \right)} \quad \forall\, w \in \mathbb{C} \setminus A.$$

Moreover, the right-hand side converges uniformly on compact sets in $\mathbb{C} \setminus A$.

Proof.- Let $\Omega = \mathbb{C} \setminus A$ so that $f : \Omega \to \mathbb{C}$ is holomorphic. Recall that for each $m \geq 1$, $C_m = C(0; R_m)$ (see Assumption 2 above).

Let N_m be the positive integer so that

$a_1, a_2, \ldots, a_{N_m} \in$ Interior (C_m) (i.e. $D(0; R_m)$) ; $a_n \in$ Exterior (C_m) $\forall\, n > N_m$



(recall $a_j \notin C_m$ for all j)

As the poles are arranged in non-decreasing modulus : $N_1 \leq N_2 \leq \cdots$

Let $w \in \Omega$. By Cauchy's integral formula (assuming $|w| < R_m$)

$$\frac{1}{2\pi i} \int_{C_m} \frac{f(z)}{z-w}\, dz = f(w) + \sum_{k=1}^{N_m} \frac{b_k}{a_k - w} \qquad -(1)$$

Now replace $\frac{1}{z-w}$ by $\frac{1}{z}\left(1 + \frac{w}{z-w}\right)$. We get (again by Cauchy's formula:

$$\frac{1}{2\pi i} \int_{C_m} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int_{C_m} \frac{f(z)}{z}\, dz + \frac{w}{2\pi i} \int_{C_m} \frac{f(z)}{z(z-w)}\, dz$$

$$= f(0) + \sum_{k=1}^{N_m} \frac{b_k}{a_k} + \frac{w}{2\pi i} \int_{C_m} \frac{f(z)}{z(z-w)}\, dz \qquad -(2)$$

Combining (1) and (2) we get

$$f(w) - \left(f(0) + \sum_{k=1}^{N_m} \left(\frac{b_k}{w-a_k} + \frac{b_k}{a_k}\right)\right) = \frac{w}{2\pi i} \int_{C_m} \frac{f(z)}{z(z-w)}\, dz \qquad -(*)$$

Now $$\left| \frac{w}{2\pi i} \int_{C_m} \frac{f(z)}{z(z-w)}\, dz \right| \leq \frac{|w|}{\cancel{2\pi}} \frac{M \cdot \cancel{2\pi} \cancel{R_m}}{\cancel{R_m}(R_m - |w|)}$$

$$= \frac{M \cdot |w|}{R_m - |w|} \to 0 \text{ as } m \to \infty.$$

Hence, $$f(w) = f(0) + \sum_{k=1}^{\infty} b_k \left(\frac{1}{w-a_k} + \frac{1}{a_k} \right).$$

Uniform convergence of $f(0) + \sum_{K=1}^{\infty} b_K \left(\frac{1}{w-a_K} + \frac{1}{a_K} \right)$:
(rel. to compact sets)

Let $K \subset \mathbb{C} \setminus A$ be a compact set, $a \in \mathbb{R}_{>0}$ s.t. $|w| < a$ $\forall w \in K$.

By $(*)$ on the previous page, for $m \in \mathbb{Z}_{\geq 1}$ sufficiently large so that $R_m > a$, we get: $(\forall w \in K)$

$$\left| \sum_{K=N_m+1}^{\infty} b_K \left(\frac{1}{w-a_K} + \frac{1}{a_K} \right) \right| = \left| \frac{w}{2\pi i} \int_{C_m} \frac{f(z)}{z(z-w)} dz \right| \leq \frac{M \cdot a}{R_m - a} .$$

As $R_m \to \infty$ $(m \to \infty)$, ~~the~~ given any $\varepsilon > 0$, we can take m large enough so that $\frac{M a}{R_m - a} < \varepsilon$. This proves the uniform convergence as claimed. □

(32.2) Remark on Assumption 2 of Theorem (32.1).

The choice of circles $C_m = C(0; R_m)$ $(m \geq 1)$ was merely for convenience of stating the result, not a necessity. In general, the proof of Theorem 32.1 carries over if we relaxed Assumption 2 as follows.

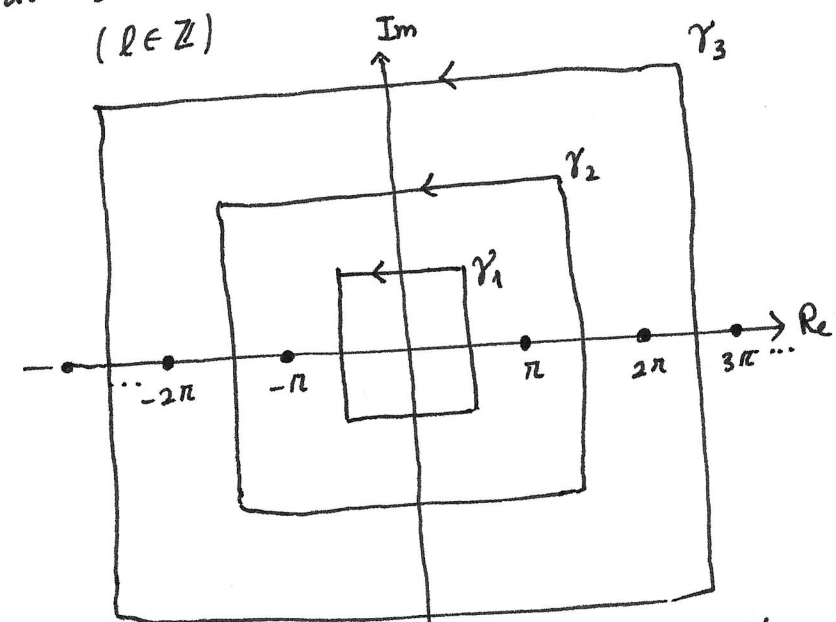
Assumption 2 revised. - For each $m \in \mathbb{Z}_{\geq 1}$, it is possible to choose a contour γ_m in $\mathbb{C} \setminus A$ so that

- $\text{Interior}(\gamma_1) \subset \text{Interior}(\gamma_2) \subset \cdots$; $\mathbb{C} = \bigcup_{m=1}^{\infty} \text{Interior}(\gamma_m)$.
  (i.e. $\forall z \in \mathbb{C}, \exists n$ s.t. $z \in \text{Interior}(\gamma_m)$ $\forall m \geq n$)
- For a fixed $z \in \mathbb{C}$, $\lim_{m \to \infty} \text{distance}(z; \gamma_m) = \infty$.
- $\exists M > 0$ s.t. $|f(z)| < M$ for every z on γ_m $(m \geq 1)$

(32.3) Example : $f(z) = \operatorname{cosec}(z) - \frac{1}{z}$.

- Poles of f $= \{ n\pi : n \in \mathbb{Z}_{\neq 0} \}$
- Residue of $\operatorname{cosec}(z)$ at $z = \ell\pi$ $(\ell \in \mathbb{Z})$ $= (-1)^\ell$ (Ex.)

Once we verify assumption 2 of Thm 32.1, we obtain the following "partial fractions" for $\operatorname{cosec}(z)$:

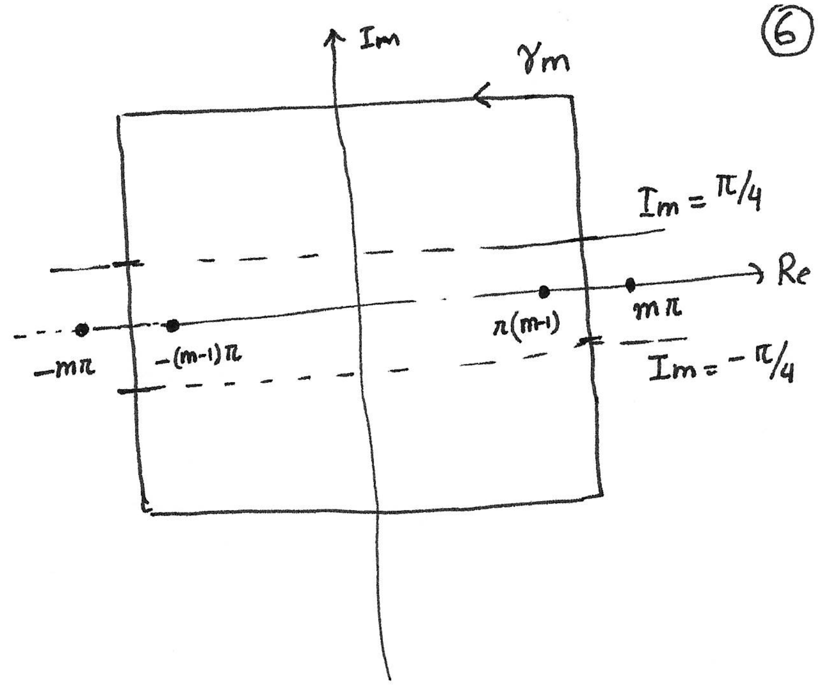


Poles of $f(z) = \operatorname{cosec}(z) - \frac{1}{z}$.

$$\boxed{\operatorname{Cosec}(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)}$$

For each $m \geq 1$, let γ_m = boundary of the square with vertices $(2m-1)\frac{\pi}{2}(\pm 1 \pm i)$ (see figure above)

We have to find a constant $M > 0$ s.t. $|f(z)| < M$ for every z on γ_m $(\forall\, m \geq 1)$.



$$\operatorname{Cosec}(z) = \frac{2i}{e^{iz}-e^{-iz}}$$

Hence, by triangle ineq.

$$|\operatorname{cosec}(z)| = \frac{2}{|e^{iz}-e^{-iz}|}$$

$$\le \frac{2}{\left||e^{iz}|-|e^{-iz}|\right|}$$

$$= \frac{2}{e^{|Im(z)|}-e^{-|Im(z)|}} < \frac{2}{e^{\pi/4}-1} \qquad \forall z \in \mathbb{C} \text{ s.t. } |Im(z)| > \frac{\pi}{4}.$$

i.e. for $z \in \gamma_m$, if $|Im(z)| > \frac{\pi}{4}$, $|\operatorname{Cosec}(z)| < \frac{2}{e^{\pi/4}-1}$.

On the segment $(2m-1)\frac{\pi}{2} + ti \quad \left(-\frac{\pi}{4} \le t \le \frac{\pi}{4}\right)$,

$$\exp\left(\left((2m-1)\frac{\pi}{2}+ti\right)\cdot i\right) = (-1)^{m-1}\, i \cdot e^{-t}$$

$$\exp\left(-\left((2m-1)\frac{\pi}{2}+ti\right)i\right) = (-1)^{m}\, i \cdot e^{+t}$$

Hence, $$|\operatorname{cosec}(z)| = \left|\frac{2\cancel{i}}{(-1)^{m-1}\cancel{i}\,(e^{t}+e^{-t})}\right| \le \frac{2}{e^{-\pi/4}+e^{-\pi/4}} = e^{\pi/4}.$$

So $$\left|\operatorname{cosec}(z) - \frac{1}{z}\right| < \boxed{\operatorname{Max}\left\{e^{\pi/4}, \frac{2}{e^{\pi/4}-1}\right\} + 1} = M.$$ $\square$

(Note: for z on γ_m, $|z| > 1$, hence $\left|\frac{1}{z}\right| < 1$).