Lecture 33

(33.0) Recall - last time we proved a "generalized partial fraction" theorem

Hypotheses: $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic ; $A \subset \mathbb{C}$ set of poles of f. $A = \{a_1, a_2, a_3, \dots\}$ arranged so that $0 < |a_1| \leq |a_2| \leq |a_3| \leq \dots$ ($0 \notin A$: assumption)

(1) We assume that every a_n is a simple pole of residue b_n.

(2) Further assume that $\forall\, m \geq 1$, we can choose a contour γ_m around 0 ; (in $\mathbb{C} \setminus A$) so that

- $\text{Interior}(\gamma_1) \subset \text{Interior}(\gamma_2) \subset \dots$
- $R_m = \text{dist}(0; \gamma_m) \to \infty$ as $m \to \infty$.
- $\exists\, C > 0$ s.t. $\text{Length}(\gamma_m) \leq C \cdot R_m \;\; \forall\, m$.
- $\exists\, M > 0$ s.t. $|f(z)| \leq M \;\; \forall\, z$ on γ_m ($\forall\, m \geq 1$).

Then
$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right) \quad \forall\, z \in \mathbb{C} \setminus A.$$

(33.1) Corollary. - (Weierstrass factorization theorem) - Let $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function ; $A = \{a_1, a_2, a_3, \dots\} \subset \mathbb{C}$ (discrete set) arranged so that $0 < |a_1| \leq |a_2| \leq \dots$. Let $m_n \in \mathbb{Z}_{\geq 1}$ be given. ; so that $g(a_n) = 0$ to order m_n. ($\forall\, n \geq 1$).

Assume (2) (Hypothesis) for $f = \frac{g'}{g}$. Then

$$g(z) = g(0) \cdot \left(\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{m_n} \cdot e^{z \cdot \frac{m_n}{a_n}} \right) \cdot e^{\frac{g'(0)}{g(0)} \cdot z}$$

Proof. – Let $f = \frac{g'}{g} : \mathbb{C} \dashrightarrow \mathbb{C}$. Then f has simple poles at A and $\operatorname{Res}_{a_n} f = m_n$. By Theorem (32.1) (Lecture 32, page 2),

$$(*) \qquad f(z) = f(0) + \sum_{n=1}^{\infty} m_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} \right)$$

For $z \in \mathbb{C} \setminus A$, let γ be a simple (piecewise smooth) path joining 0 to z and set $h(z) = \int_\gamma f(z)\,dz$. Since $f = \frac{d}{dz} \log(g) = \frac{g'}{g}$, $\int_\gamma f(z)\,dz$ (modulo $2\pi i \mathbb{Z}$) does not depend on γ.

Hence $G(z) = \exp \int_\gamma f(z)\,dz = \exp \int_\gamma \frac{g'(z)}{g(z)}\,dz$

is well-defined. Since R.H.S. of $(*)$ is uniformly convergent - termwise integral is permissible, and we get

$$G(z) = e^{\frac{g'(0)}{g(0)} \cdot z} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{m_n} e^{\frac{m_n}{a_n} \cdot z} .$$

Note that $G'(z) = G(z) \cdot \frac{g'(z)}{g(z)}$ and $g'(z) = g(z) \cdot \left(\frac{g'(z)}{g(z)}\right)$

$\Rightarrow \frac{d}{dz}\left(\frac{G(z)}{g(z)}\right) = 0$; i.e. $C \cdot G(z) = g(z)$

$C = g(0)$. The result follows. $\square$

(33.2) Mittag-Leffler Theorem

Assume that a discrete set $A = \{a_1, a_2, a_3, \dots\} \subset \mathbb{C}$ $(0 < |a_1| < |a_2| < \dots)$ is given and $\forall n \geq 1$, we have a polynomial $P_n(z)$. Then $\exists$ a mero. fn. $f : \mathbb{C} \dashrightarrow \mathbb{C}$ s.t. f has poles exactly at A and the Laurent Series expansion of f near a_n is of the form:

$$f(z) = P_n\left(\frac{1}{z-a_n}\right) + \underset{(\text{near } a_n)}{\text{holomorphic part}}$$

Any two such functions differ by a holomorphic, <u>entire</u>, function.

Example : (1) Let $A = \{n\pi : n \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{C}$. Let $P_n(z) = z$.

The theorem guarentees the existence of a mero. fn. $f : \mathbb{C} \dashrightarrow \mathbb{C}$ with poles at $n\pi$, s.t. $f(z) = \dfrac{1}{z-n\pi} + \underset{\substack{(\text{dep. on } n) \\ (\text{i.e. near } n\pi)}}{\text{holomorphic part}}$

i.e. f has simple pole at $n\pi$, of residue 1.

One could try $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z-n\pi}$, but this series is divergent

However, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$ does give a convergent series

(Ex. this series converges uniformly for $z \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}\pi$ and (rel. to compact sets).

hence defines a mero. fn. $\mathbb{C} \setminus \mathbb{Z}_{\geq 1}\pi \to \mathbb{C}$.)

So, we can define $f(z) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$.

(2). Assume that $A = \{\sqrt{n} : n \in \mathbb{Z}_{\geq 1}\}$. $P_n(z) = z$ $(\forall n \geq 1)$.

Again Theorem $\Rightarrow$ $\exists$ a mero fn. $f : \mathbb{C}$ ~~\A~~ $\dashrightarrow \mathbb{C}$ with simple poles at $\sqrt{n}$ with residue n.

As before $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z-\sqrt{n}}$ is divergent. The "fix" from previous part $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}}\right)$ still gives a divergent series.

(Reason: $\displaystyle\frac{1}{z-a} = -\frac{1}{a}\cdot\frac{1}{1-\frac{z}{a}} = -\sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}$ for $|a|>|z|$.

So, for $N > |z|$,
$$\sum_{n=N}^{\infty}\frac{1}{z-\sqrt{n}} = -\sum_{n=N}^{\infty}\left(\sum_{l=0}^{\infty}\frac{z^l}{(\sqrt{n})^{l+1}}\right)$$
$$= -\sum_{l=0}^{\infty} z^l\left(\sum_{l=0}^{\infty}\frac{1}{(\sqrt{n})^{l+1}}\right)$$
$\uparrow$ may be divergent

$$\sum_{n=N}^{\infty}\left(\frac{1}{z-\sqrt{n}}+\frac{1}{\sqrt{n}}\right) = -\sum_{l=1}^{\infty} z^l\left(\sum_{l=0}^{\infty}\frac{1}{(\sqrt{n})^{l+1}}\right)$$
$\uparrow$ still divergent for $l=1$.

$$\rightsquigarrow \quad f(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{z}{(\sqrt{n})^2}\right) \text{ works.}$$

(Ex. If $\displaystyle\lim_{n\to\infty}\frac{a_n^2}{n} = c \in \mathbb{R}_{>0}$, then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-a_n}+\frac{1}{a_n}+\frac{z}{a_n^2}\right)$ converges uniformly on $\mathbb{C}\setminus\{a_n\}$).
(rel. to cpct sets)

(33.3) Proof of Thm. (33.2). For each $n \geq 1$, $P\left(\frac{1}{z-a_n}\right)$ is holomorphic near $z=0$. Using Taylor series expansion near $z=0$, we can find polynomials $c_n(z)$ s.t.

$$(*):\quad \left| P_n\left(\frac{1}{z-a_n}\right) - c_n(z) \right| < \frac{1}{2^n}. \quad \forall\, z \text{ s.t. } |z| < \frac{|a_n|}{2}$$

(Note: $P_n\left(\frac{1}{z-a_n}\right)$ is holomorphic $\mathbb{C} \setminus \{a_n\} \to \mathbb{C}$ - thus radius of convergence of its Taylor series expansion near 0 is $|a_n|$.

If $$P_n\left(\frac{1}{z-a_n}\right) = \sum_{l=0}^{\infty} d_{n;l}\, z^l \quad (|z| < |a_n|)$$ ← Taylor series of $P_n\left(\frac{1}{z-a_n}\right)$ near $z=0$.

then, by uniform convergence of the Taylor series, given $\varepsilon > 0$ ($\varepsilon = \frac{1}{2^n}$ in our case), and $r < |a_n|$ ($r = \frac{|a_n|}{2}$ in our case), we can find $N > 0$ s.t.

$$\left| P_n\left(\frac{1}{z-a_n}\right) - \sum_{l=0}^{N} d_{n;l}\, z^l \right| < \varepsilon \quad \text{for every } z \text{ s.t. } |z| \leq r.)$$

← polynomial in z.

Claim: $$\sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - c_n(z) \right)$$ converges uniformly (on compact sets in $\mathbb{C} - A$).

(Hence defines a holomorphic function $f: \mathbb{C} \setminus A \to \mathbb{C}$.)

Proof of the claim: Let $K \subset \mathbb{C} \setminus A$ compact and $\varepsilon > 0$ be given.

Let $R > 0$ be such that $|z| \leq R$ $\forall\, z \in K$.

Choose $N > 0$ large enough so that $$\begin{cases} |a_n| > 2R & \forall\, n \geq N \\ 2^N > \frac{1}{\varepsilon} \end{cases}$$

(note: for every $z \in K$ and $n \geq N$; $|z| \leq R < \frac{|a_n|}{2}$, hence $(*)$ is valid.)

Then, for every $n > N$ and $m \geq 0$, $z \in K$, we have:

$$\left| \sum_{k=n}^{n+m} \left(P_k\left(\frac{1}{z-a_k}\right) - c_k(z) \right) \right| \leq \sum_{k=n}^{n+m} \left| P_k\left(\frac{1}{z-a_k}\right) - c_k(z) \right|$$

$$\leq \sum_{k=n}^{n+m} \frac{1}{2^k} < \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \varepsilon.$$

Finally - if $f_1, f_2 : \mathbb{C} \setminus A \to \mathbb{C}$ are two holomorphic functions satisfying the condition imposed in Theorem (33.2) (i.e. $\forall n \geq 1$, Laurent series expansions of both f_1 and f_2 near $z = a_n$ have the same singular part : $P_n\left(\frac{1}{z-a_n}\right)$.)

then $f_1 - f_2$ has removable singularities at A - i.e.

$f_1 - f_2 : \mathbb{C} \to \mathbb{C}$ is <u>an entire holomorphic function.</u>

(33.4) Corollary of Thm (33.2) (same proof as in Cor 33.1): [$A = \{a_1, a_2, \dots\} \subset \mathbb{C}$ as before] Let $m_n \in \mathbb{Z}_{\geq 1}$ be given, for each $n \geq 1$. Then $\exists\ g : \mathbb{C} \to \mathbb{C}$ holomorphic ; s.t. g has zeroes at (and only at) a_n of multiplicity m_n ($\forall n \geq 1$).

If $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ are two such functions, then $g_1 = g_2 \cdot e^h$ for some holomorphic function $h : \mathbb{C} \to \mathbb{C}$.