Lecture 34

(34.0) Euler's gamma function: $\Gamma(z)$ was defined by Euler in 1729, as solution to the following problem posed by Goldbach:

Find $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $f(n) = n!$ $\forall\, n \in \mathbb{Z}_{\geq 0}$.

Euler's definition was inspired by the following calculation:

Ex. $\displaystyle\int_0^\infty t^n \cdot e^{-t}\, dt = n! \quad (\forall\, n \in \mathbb{Z}_{\geq 0})$.

Proof. - $(n=0)$ $\displaystyle\int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_{t=0}^{t=\infty} = 1 - \lim_{t\to\infty} e^{-t} = 1.$

For $n \geq 0$: $\displaystyle\int_0^\infty t^{n+1} e^{-t}\, dt = \left[t^{n+1}\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} + (n+1)\int_0^\infty t^n e^{-t}\, dt$

(integration by parts)

$\displaystyle = (n+1)\int_0^\infty t^n e^{-t}\, dt = (n+1)\cdot n!$ (induction)

$= (n+1)!$ $\square$

Euler suggested to define $\lfloor x := \displaystyle\int_0^\infty t^x e^{-t}\, dt$ $(x \in \mathbb{R}_{>-1})$.

(old notation for factorial)

The notation $\Gamma(z)$ and a shift in x are due to Lagrange:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt.$$

We will see a proof that the integral defined above converges uniformly rel. to compact subsets of $\{z \mid Re(z) > 0\}$ and hence defines a holomorphic function $\Gamma : \{Re(z) > 0\} \to \mathbb{C}$.

(34.1) <u>Holomorphic functions defined by integrals</u>. Consider, more generally,

- $\Omega \subset \mathbb{C}$ open, connected set.
- $G(t,z) : \mathbb{R}_{>0} \times \Omega \to \mathbb{C}$

$\rightsquigarrow$ $g : \Omega \to \mathbb{C}$ defined by

$$g(z) = \int_0^\infty G(t,z)\, dt.$$

Sufficient conditions on $G(t,z)$ so that $g : \Omega \to \mathbb{C}$ is a holomorphic function., are given in the following theorem.

<u>Theorem.</u> Assume that (1) $G : \mathbb{R}_{>0} \times \Omega \to \mathbb{C}$ is continuous.

(2) $\forall\, t \in \mathbb{R}_{>0}$; $G(t,-) : \Omega \to \mathbb{C}$ is holomorphic.

(3) $\partial_z G : \mathbb{R}_{>0} \times \Omega \to \mathbb{C}$ is continuous

(4) $\displaystyle\int_0^\infty G(t,z)\, dt := \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R G(t,z)\, dt$ - converges

uniformly on compact sets in Ω (see Remark below)

Then $g(z) := \displaystyle\int_0^\infty G(t,z)\, dt$ is a holomorphic function of $z \in \Omega$ and

$$g'(z) = \int_0^\infty \partial_z G(t,z)\,dt .$$

<u>Remark on assumption (4) of the theorem.</u> The infinite integral

$$\int_0^\infty G(t,z)\,dt \quad \text{is defined as the limit} \quad \int_r^R G(t,z)\,dt \quad \text{as } \begin{matrix} r \to 0 \\ R \to \infty \end{matrix} .$$

By Cauchy's criterion - the existence of this limit means:

$$\forall \varepsilon > 0 \; ; \; \exists\, r > 0,\ R > 0 \quad \text{such that} \quad \left| \int_{r_1}^{r_2} G(t,z)\,dt \right| < \varepsilon$$

$$\left| \int_{R_1}^{R_2} G(t,z)\,dt \right| < \varepsilon$$

for every $0 < r_1 < r_2 < r$ and $R < R_1 < R_2$. Assumption (4) means that these r and R can be chosen for all $z \in K$ (a compact subset of Ω). That is: for every $K \subset \Omega$ (compact) and $\varepsilon > 0$, $\exists\, r > 0, R > 0$ so that

$$\left| \int_{r_1}^{r_2} G(t,z)\,dt \right| < \varepsilon \quad ; \quad \left| \int_{R_1}^{R_2} G(t,z)\,dt \right| < \varepsilon$$

$$\forall \quad 0 < r_1 < r_2 < r \; ; \quad R < R_1 < R_2 \; ; \quad z \in K.$$

<u>In particular</u>: choose $a_1 \geq a_2 \geq \dots \quad \lim_{n\to\infty} a_n = 0$ and

$$b_1 \leq b_2 \leq \dots \quad \lim_{n\to\infty} b_n = \infty$$

set $g_n(z) = \displaystyle\int_{a_n}^{b_n} G(t,z)\,dt$. Then $\{g_n : \Omega \to \mathbb{C}\}_{n \geq 1}$ is a <u>uniformly convergent sequence of functions</u>.
(rel. to compact sets in Ω)

(34.2) <u>Proof of Theorem (34.1)</u>. Let $a_1 \geq a_2 \geq \cdots \to 0$ as $n \to \infty$
$b_1 \leq b_2 \leq \cdots \to \infty$

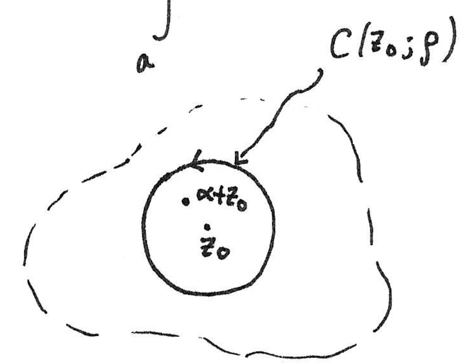
be as above and $g_n(z) := \displaystyle\int_{a_n}^{b_n} G(t,z)\,dt$ $(n \geq 1)$. Since $\{g_n\}_{n \geq 1}$ is uniformly convergent $g(z) = \lim\limits_{n \to \infty} g_n(z)$ $(\forall z \in \Omega)$, by Weierstrass' Theorem (19.6), it is enough to show that each g_n is holomorphic and $g_n'(z) = \displaystyle\int_{a_n}^{b_n} \partial_z G(t,z)\,dt$.

So, let $[a,b] \subset \mathbb{R}_{>0}$ be fixed and let $h(z) = \displaystyle\int_a^b G(t,z)\,dt$.

Let $z_0 \in \Omega$ and $\rho > 0$ be such that

$$\overline{D}(z_0;\rho) \subset \Omega.$$

$C_\rho = C(z_0;\rho) =$ circle of radius ρ centered at z_0.



Since for each fixed $t \in \mathbb{R}_{>0}$, $G(t,-) : \Omega \to \mathbb{C}$ is holomorphic, by Cauchy's integral formula, we get

$$h(z_0) = \int_a^b G(t,z_0)\,dt = \frac{1}{2\pi i}\int_a^b\left(\int_C \frac{G(t,z)}{z-z_0}\,dz\right)dt$$

The remainder of the proof follows along the same lines as that of Cauchy's integral formula (13.3).

<u>Continuity of h.</u> $\quad h(z_0+\alpha) - h(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_a^b\int_C G(t,z)\left(\frac{1}{z-z_0-\alpha} - \frac{1}{z-z_0}\right)dz\,dt$

$(|\alpha| < \rho)$

$$= \frac{\alpha}{2\pi i}\int_a^b\int_C \frac{G(t,z)}{(z-z_0)(z-z_0-\alpha)}\,dz\,dt$$

Let $M = \mathrm{Max}\left\{|G(t,z)| : (t,z) \in \underbrace{[a,b]\times C}_{\text{cpct}}\right\}$. Then, as before, we can bound

$$|h(z_0+\alpha) - h(z_0)| \leqslant \frac{|\alpha|}{\cancel{2\pi 1}}\,\frac{M}{\rho(\rho-|\alpha|)}\cdot\cancel{2\pi\rho}\cdot(b-a)$$

So, given $\varepsilon > 0$; let $r > 0$ be (say $< \frac{\rho}{2}$) such that

$$\frac{2M(b-a)}{\rho}\cdot r < \varepsilon.$$

Then $\forall\, \alpha \in \mathbb{C}$ so that $|\alpha| < r$, we get

$$|h(z_0+\alpha) - h(z_0)| \leqslant |\alpha|\cdot\frac{M(b-a)}{\rho-|\alpha|} < \frac{r\cdot M\cdot(b-a)}{\rho - \rho/2} < \varepsilon.$$

hence h is continuous at z_0 $(\forall\, z_0 \in \Omega)$.

<u>Complex differentiability of h</u> : z_0, ρ, α as above. We get

$$\frac{h(z_0+\alpha) - h(z_0)}{\alpha} = \frac{1}{2\pi i} \int_a^b \int_C \frac{G(t,z)}{(z-z_0)(z-z_0-\alpha)} \, dz \, dt$$

<u>Claim</u> (exactly same as the claim from Lecture 13 - pages 5 - proof of Cauchy's integral formula):

$$\lim_{\alpha \to 0} \frac{h(z_0+\alpha) - h(z_0)}{\alpha} = \frac{1}{2\pi i} \int_a^b \int_C \frac{G(t,z)}{(z-z_0)^2} \, dz .$$

See Lecture 13, pages 5-6 for a proof. $\square$

(34.3) For $\Gamma(z)$, we consider $G(t,z) = t^{z-1} e^{-t}$ $\left(\begin{array}{l} t \in \mathbb{R}_{>0} \\ z \in \Omega \end{array} \right)$

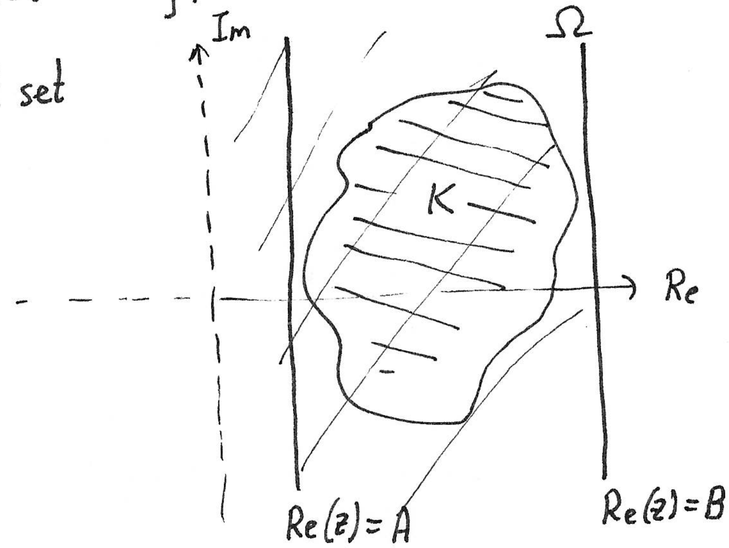
$$= e^{(z-1)\ln(t)} \cdot e^{-t}$$

Ω: $\{ w \mid \mathrm{Re}(w) > 0 \}$

<u>Lemma</u>. Assumption (4) of Theorem (34.1) holds for

$$G(t,z) = t^{z-1} e^{-t} : \mathbb{R}_{>0} \times \Omega \to \mathbb{C} ,$$

where $\Omega = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 0 \}$.



Proof.- Let $K \subset \Omega$ be a compact set and $\varepsilon > 0$ be given.

Let $0 < A < B$ be s.t.

$$A \leq \mathrm{Re}(z) \leq B \quad \forall z \in K .$$

We need to prove: (1) $\exists\, r>0$ s.t. $\left| \int_{r_1}^{r_2} t^{z-1} e^{-t}\, dt \right| < \varepsilon$

for every $0 < r_1 < r_2 < r$; $z \in K$.

Proof. Note that for $t \in (0,1)$, $\ln(t)$ is negative, hence

$$|t^{z-1}| = \left| e^{(z-1)\ln(t)} \right| = e^{Re((z-1)\ln(t))}$$
$$\leq e^{(A-1)\ln(t)} = t^{A-1} \quad \left(\begin{array}{c} \forall t \in (0,1) \\ z \in K \end{array} \right).$$

So, choose $r \in (0,1)$ s.t. $r^A < \varepsilon A$. Then, $\forall\, 0 < r_1 < r_2$; $z \in K$,

we have
$$\left| \int_{r_1}^{r_2} t^{z-1} e^{-t}\, dt \right| \leq \int_0^r t^{A-1} dt = \frac{r^A}{A} < \varepsilon.$$

(2) : $\exists\, R>0$ s.t. $\left| \int_{R_1}^{R_2} t^{z-1} e^{-t}\, dt \right| < \varepsilon$, for every $R < R_1 < R_2$, $z \in K$.

Proof. For $t > 1$, $\ln(t) > 0$ and hence $|t^{z-1}| = e^{Re((z-1)\ln(t))} \leq t^{B-1}$
$(\forall t \in (1,\infty)$; $z \in K)$.

Since $t^{B-1} \cdot e^{-t/2} \to 0$ as $t \to \infty$; $\exists\, t_0 \in \mathbb{R}_{>0}$ s.t.

$t^{B-1} e^{-t} \leq 1$ $\forall t \geq t_0$. Pick $R > t_0$ s.t. $e^{-R/2} < \frac{\varepsilon}{2}$. Then :

$\forall\, R_1, R_2 > R$; $z \in K$:

$$\left| \int_{R_1}^{R_2} t^{z-1} e^{-t}\, dt \right| \leq \int_R^\infty \underbrace{\left(t^{B-1} e^{-t/2}\right)}_{(\leq 1)} e^{-t/2}\, dt \leq \int_R^\infty e^{-t/2}\, dt = 2 \cdot e^{-R/2} < \varepsilon.$$

$\square$