(35.0) Recall that we defined $\Gamma(z)$, for $z \in \Omega = \{Re > 0\}$ (right-half plane),

as: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$. By the calculation from §34.0,

$\Gamma(n) = (n-1)! \quad \forall\, n \in \mathbb{Z}_{\geq 1}$.

(35.1) Difference equation for $\Gamma(z)$.

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

Proof (integration by parts). $\Gamma(z+1) = \int_0^\infty t^{z} e^{-t}\, dt$

$$= \left[t^{z} \frac{e^{-t}}{-1} \right]_{t=0}^{t=\infty} + \int_0^\infty z \cdot t^{z-1} e^{-t}\, dt$$

$= z \cdot \Gamma(z)$ (Since, for $Re(z) > 0$, $\lim_{t \to \infty} t^{z} e^{-t} = 0$, $\lim_{t \to 0} t^{z} = 0$) □

(35.2) Extending the domain to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Using $\Gamma(z+1) = z\,\Gamma(z)$, we can extend the domain of definition of $\Gamma(z)$ from $\{Re(z) > 0\}$ to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Let $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Let $N \in \mathbb{Z}_{\geq 0}$ be such that $\mathrm{Re}(z+N) > 0$. Thus $\Gamma(z+N) = \int_0^\infty t^{z+N-1} \cdot e^{-t} dt$ is defined

Set $\Gamma(z) = \dfrac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}$. Thus Γ becomes a meromorphic function $\Gamma : \mathbb{C} \dashrightarrow \mathbb{C}$ with poles at $\{0, -1, -2, \cdots\}$.

(35.3) Poles of Γ are all simple (i.e., order of the pole at $z = -n$ is 1, $\forall\, n \in \mathbb{Z}_{\geq 0}$).

$$\operatorname*{Res}_{z=0} \Gamma(z) = \operatorname*{Res}_{z=0} \frac{\Gamma(z+1)}{z} = \lim_{z \to 0} \Gamma(z+1) = \Gamma(1) = 1.$$

$$\operatorname*{Res}_{z=-N} \Gamma(z) = \operatorname*{Res}_{w=0} \Gamma(w-N) = \operatorname*{Res}_{w=0} \frac{\Gamma(w)}{(w-1)\cdots(w-N)}$$

$$\left((w-1)\cdots((w-N)\Gamma(w-N)) = (w-1)\cdots(w-N+1)\Gamma(w-N+1) = \cdots = \Gamma(w) \right)$$

$$= \frac{1}{(-1)(-2)\cdots(-N)} = \frac{(-1)^N}{N!}.$$

(35.4) $\underline{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$. $\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} \cdot e^{-t} dt$

Set $u = t^{1/2}$ or $t = u^2$ so ~~th~~ that $dt = 2u\,du$.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty \frac{1}{\cancel{u}} e^{-u^2} 2\cancel{u}\, du = \int_{-\infty}^{\infty} e^{-u^2} du \quad \left(e^{-x^2} \text{ is even, so } \int_{-\infty}^{\infty} = 2 \cdot \int_0^\infty \right)$$

[Gaussian integral]

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \int_{-\infty}^{\infty} e^{-u_1^2}\,du_1 \int_{-\infty}^{\infty} e^{-u_2^2}\,du_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u_1^2+u_2^2)}\,du_1\,du_2$$

Change to polar coordinates $u_1 = r\cos(\theta)$, $u_2 = r\sin(\theta)$
$(r \in (0,\infty),\ \theta \in [0,2\pi])$

$du_1\,du_2 \rightsquigarrow r\,dr\,d\theta$.

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = [\theta]_0^{2\pi}\left[\frac{e^{-r^2}}{-2}\right]_0^{\infty} = 2\pi\cdot\frac{1}{2}$$

$$= \pi. \quad \Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

(35.5) Beta function. Let $p, q \in \mathbb{R}_{>0}$ and consider

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad -\ (1)$$

Or, changing $x = \cos^2(\theta)$, $dx = -2\sin(\theta)\cos(\theta)\,d\theta$; (1) becomes

$$B(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}(\theta)\,\sin^{2q-1}(\theta)\,d\theta \qquad -\ (2)$$

Euler computed this integral as:

$$\boxed{B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}}$$

Note: we recover the computation of §35.4 by setting $p=q=\frac{1}{2}$ in (2)

Proof. - $\Gamma(p)\,\Gamma(q) = \int_0^\infty\int_0^\infty e^{-(t_1+t_2)} \cdot t_1^{p-1}\, t_2^{q-1}\, dt_1\, dt_2$

Change of variables : $x = t_1 + t_2 \qquad (0 < x < \infty)$

$$y = \frac{t_1}{t_1+t_2} \qquad (0 < y < 1)$$

Or, equivalently : $t_1 = xy$ and $t_2 = x - t_1 = x(1-y)$

$dt_1\, dt_2 \rightsquigarrow x\, dx\, dy$ (verify this!)

$$\Rightarrow \Gamma(p)\,\Gamma(q) = \int_0^1\int_0^\infty e^{-x}\,(xy)^{p-1}\, x^{q-1} \cdot (1-y)^{q-1}\, x\, dx\, dy$$

$$= \int_0^\infty e^{-x}\, x^{p+q-1}\, dx \cdot \int_0^1 y^{p-1}(1-y)^{q-1}\, dy$$

$$= \Gamma(p+q) \cdot B(p,q) \qquad \square$$

(35.6) Gamma function as another uniform limit (Euler).

<u>Theorem.</u> For $N \in \mathbb{Z}_{\geq 1}$, let $G_N(z) = \int_0^N t^{z-1}\left(1 - \frac{t}{N}\right)^N dt$

(for $\mathrm{Re}(z) > 0$).

(1) $\{G_N(z)\}_{N\geq 1}$ converges uniformly (rel. to compact sets) in $\Omega = \{z \mid \mathrm{Re}(z) > 0\}$.

(2) $$G_N(z) = \frac{N!}{z(z+1)\cdots(z+N)} \cdot N^z \quad (\forall\, z \in \Omega)$$

Remark. - The functions $\dfrac{N!}{z(z+1)\cdots(z+N)} N^z$ are defined on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

We will see below that the convergence of this sequence of functions is uniform (rel. to cpct sets) in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and hence defines a holomorphic function (still denoted by Γ) $\mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$.

Proof. - (2): $$G_N(z) = \int_0^N t^{z-1}\left(1 - \frac{t}{N}\right)^N dt$$

$$= \int_0^1 N^{z-1} \cdot \tau^{z-1} (1-\tau)^N \cdot N \cdot d\tau \quad \left(\text{change of variables } t = N\tau\right)$$

$$= N^z \cdot \int_0^1 \tau^{z-1} (1-\tau)^N d\tau = N^z B(z, N+1)$$

(see §35.5)

$$= N^z \frac{\Gamma(z)\,\Gamma(N+1)}{\Gamma(z+N+1)}$$

$$= N^z \frac{N!}{z(z+1)\cdots(z+N)} \quad \left(\begin{array}{l}\text{since } \Gamma(N+1) = N! \\ \text{and } \Gamma(w+1) = w\,\Gamma(w)\end{array}\right)$$

(1) Consider the difference $\Gamma(z) - G_N(z)$

$$\int_0^\infty t^{z-1} e^{-t}\, dt - \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$$

$$= \int_N^\infty t^{z-1} e^{-t}\, dt + \int_0^N t^{z-1}\left(e^{-t} - \left(1-\frac{t}{N}\right)^N\right) dt$$

Inequality : $0 \leqslant e^{-t} - \left(1-\frac{t}{N}\right)^N \leqslant e^{-t}\frac{t^2}{N}$. $(0 \leqslant t \leqslant N)$.

(Proof of the inequality : Note $e^x = 1 + x + \frac{x^2}{2!} + \cdots > 1+x \quad (\forall x > 0)$

$< 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad (\forall x \in (0,1))$

i.e. $1 + x \leqslant e^x \leqslant \frac{1}{1-x} \quad \forall\, 0 \leqslant x \leqslant 1.$

Taking the inverse : $1 - x \leqslant e^{-x} \leqslant \frac{1}{1+x}$

Set $x = \frac{t}{N}$; and raise to N-th power : $\left(1-\frac{t}{N}\right)^N \leqslant e^{-t} \quad (0 \leqslant t \leqslant N)$

$\left(1+\frac{t}{N}\right)^N \leqslant e^{t}$

$$\Rightarrow \quad e^{-t} - \left(1-\frac{t}{N}\right)^N = e^{-t}\left(1 - e^{t}\left(1-\frac{t}{N}\right)^N\right)$$

$$\leqslant e^{-t}\left(1 - \left(1+\frac{t}{N}\right)^N\left(1-\frac{t}{N}\right)^N\right)$$

$$= e^{-t}\left(1 - \left(1-\frac{t^2}{N^2}\right)^N\right) \leqslant e^{-t}\frac{t^2}{N} \quad \text{using}$$

$(1-y)^n \geqslant 1 - ny \quad \forall\, n \geqslant 0,\ y \in (0,1)$ – easy exercise by induction on n.)

Thus $\Gamma(z) - G_N(z) = \int_N^\infty t^{z-1} e^{-t}\,dt + \int_0^N t^{z-1}\left(e^{-t} - \left(1-\frac{t}{N}\right)^N\right) dt$

Let K be a compact set ; $K \subset \Omega$, and $\varepsilon > 0$ be given. Let $B \in \mathbb{R}_{>0}$ be such that $Re(z) < B \quad \forall z \in K$.

- Since $\int_0^\infty t^{z-1} e^{-t}\,dt$ converges uniformly, $\exists N_1 \gg 0$ s.t.

$$\left| \int_{N_1'}^\infty t^{z-1} e^{-t}\,dt \right| < \frac{\varepsilon}{2} \quad \forall z \in K, \; N_1' > N_1$$

(see Lecture 34, page 7)

- $$\left| \int_0^N t^{z-1}\left(e^{-t} - \left(1-\frac{t}{N}\right)^N\right) dt \right| \leq \int_0^N t^{B-1} \cdot \frac{t^2 e^{-t}}{N}\,dt$$

$$= \frac{\Gamma(B+2)}{N}$$

Choose $N_2 > 0$ so that $\frac{\Gamma(B+2)}{N_2} < \frac{\varepsilon}{2}$.

Hence for every $N > \mathrm{Max}\{N_1, N_2\}$, $z \in K$, we have

$$\left| \Gamma(z) - G_N(z) \right| < \varepsilon. \qquad \square$$