Lecture 36

(36.0) Recall: $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt$ $\quad (\operatorname{Re}(z) > 0)$. We proved in the previous lecture:

(i) $\Gamma(z+1) = z\,\Gamma(z)$

(ii) $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ $\quad (p,q \in \mathbb{R}_{>0})$

(iii) $\Gamma(z) = \lim_{N\to\infty} \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$ uniformly rel. to cpct sets in $\{\operatorname{Re}(z)>0\}$.

$$G_N(z) = \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt = N^z \cdot \frac{N!}{z(z+1)\cdots(z+N)}$$

(iv) By Theorem (34.1), $\Gamma'(z) = \int_0^\infty \ln(t)\, t^{z-1} e^{-t}\, dt$.

(36.1) Euler-Mascheroni constant. $\gamma = 0.5772\ldots$

Consider the logarithmic derivative of $G_N(z) = N^z \dfrac{N!}{z(z+1)\cdots(z+N)}$:

$$\frac{G_N'(z)}{G_N(z)} = \ln(N) - \sum_{k=0}^N \frac{1}{z+k}. \qquad \frac{G_N'(1)}{G_N(1)} = \ln(N) - 1 - \frac{1}{2} - \cdots - \frac{1}{N}.$$

$$\Rightarrow \Gamma'(1) = \lim_{N\to\infty}\left(\ln(N) - \sum_{n=1}^N \frac{1}{n}\right).$$

Define $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^N \frac{1}{n} - \ln(N)\right)$: Euler-Mascheroni constant.

Direct proof that $\lim_{N\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{N}-\ln(N)\right)$ exists.

Let $u_n = \int_0^1 \frac{t}{n(t+n)}\,dt = \int_0^1\left(\frac{1}{n}-\frac{1}{t+n}\right)dt = \frac{1}{n}-\ln(n+1)+\ln(n).$

$$\Rightarrow \quad \sum_{n=1}^{N} u_n = 1+\frac{1}{2}+\dots+\frac{1}{N}-\ln(N+1)$$

Since $u_n < \int_0^1 \frac{1}{n^2}\,dt = \frac{1}{n^2}$, and $u_n \geq 0$; $\left\{\sum_{n=1}^{N} u_n\right\}$ is an increasing sequence of positive real numbers, bounded above by $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$; hence convergent.

Note - since $\lim_{N\to\infty}\left(\ln(N+1)-\ln(N)\right)=0$,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln(N)\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln(N+1)\right). \qquad \square$$

(36.2) Weierstrass' formula for $\Gamma(z)$.

Let $$\Gamma_1(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^{-1}\cdot e^{z/n}\right\}$$

Theorem. (1) $\left\{\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}\right\}_{N=1}^{\infty}$ converges uniformly

(rel. to compact sets in $\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$).

(2) $$\Gamma_1(z) = \lim_{N\to\infty} N^z \frac{N!}{z(z+1)\dots(z+N)} = \Gamma(z). \quad \forall z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$$

Proof (2): $$G_N(z) = N^z \frac{N!}{z(z+1)\cdots(z+N)} = \frac{e^{z\cdot \ln(N)}}{z} \frac{1}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{N}\right)}$$

$$= \frac{1}{z} \cdot \prod_{n=1}^{N} \left(\left(1+\frac{z}{n}\right)^{-1} \cdot e^{z/n} \right) \cdot \underbrace{e^{z\left(\ln(N) - 1 - \frac{1}{2} - \dots - \frac{1}{N}\right)}}_{\to -\gamma \text{ as } N\to\infty}$$

$$\Rightarrow \Gamma(z) = \lim_{N\to\infty} G_N(z) = \frac{1}{z} \cdot e^{-\gamma z} \cdot \prod_{n=1}^{\infty} \left(\left(1+\frac{z}{n}\right)^{-1} \cdot e^{z/n} \right) = \Gamma_1(z).$$

(1) For the sake of the proof, let $H_N = z \cdot \prod_{n=1}^{N} \left(1+\frac{z}{n}\right) e^{-z/n}$.

We begin by making some crude estimates. Assume $K \subset \mathbb{C}$ compact set is given, and let $N_0 \in \mathbb{Z}_{\geq 1}$ be such that $|z| < \frac{N_0}{2}$ $\forall z \in K$.

Then, for ~~N~~ $n > N_0$, $\log\left(1+\frac{z}{n}\right) = \frac{z}{n} - \frac{z^2}{2n^2} + \frac{z^3}{3\cdot n^3} - \dots$ is well-defined

and $$\left| \log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right| = \left| -\frac{z^2}{2n^2} + \frac{z^3}{3\cdot n^3} - \dots \right|$$

$$\leq \frac{|z|^2}{2n^2} \cdot \left| 1 - \frac{2z}{3\cdot n} + \frac{2z^2}{4\cdot n^2} - \dots \right|$$

$$\leq \frac{N_0^2}{4} \frac{1}{2n^2} \left(1 + \frac{1}{2} + \frac{1}{4} - \dots\right)$$

$$= \frac{N_0^2}{4n^2}.$$

$\left(\text{since } \left|\frac{2z^k}{(k+2)n^k}\right| < 1 \cdot \left(\frac{|z|}{n}\right)^k < \frac{1}{2}\right)$

Note : $$H_N(z) = \left\{ z \cdot \prod_{n=1}^{N_0} \left(1+\frac{z}{n}\right) e^{-z/n} \right\} \cdot \exp\left(\sum_{n=N_0+1}^{N} \log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right).$$

Thus, it suffices to prove the uniform convergence (for $z \in K$) of
$$\left\{ \sum_{n=N_0+1}^{N} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right) \right\}_{N > N_0}$$. So, let $\varepsilon > 0$ be given

Choose $N_1 > N_0$ s.t. $\displaystyle\sum_{n=N_1}^{\infty} \frac{1}{n^2} < \frac{4\varepsilon}{N_0^2}$. (exists since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent)

$\forall N \geq N_1$, $l \geq 1$:
$$\left| \sum_{n=N+1}^{N+l} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right) \right| \leq \sum_{n=N+1}^{N+l} \left| \log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right|$$
$$\leq \frac{N_0^2}{4} \cdot \sum_{n=N_1}^{\infty} \frac{1}{n^2} < \varepsilon.$$

$\square$

(36.3) Relation between Γ and sine functions.

Recall : $$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$$= \prod_{m=1}^{\infty} \left(1+\frac{z}{m}\right) e^{-z/m} \cdot \prod_{m=1}^{\infty} \left(1 + \frac{(-z)}{m}\right) e^{-\frac{(-z)}{m}}$$

Using Weierstrass formula : $\dfrac{1}{\Gamma(z)} = z \cdot e^{\gamma z} \cdot \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$

We get :

$$\frac{\sin(\pi z)}{\pi \cancel{z}} = \frac{1}{\cancel{z} \cdot \Gamma(z) \cdot \cancel{e^{\gamma z}}} \cdot \frac{1}{(-z)\Gamma(-z) \cdot \cancel{e^{-\gamma z}}}$$

$$= \frac{1}{\Gamma(z)\Gamma(1-z)} \qquad \left(\text{using } \Gamma(1+w) = w\,\Gamma(w)\right)$$

Hence we have shown : $$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = \pi \cdot \operatorname{cosec}(\pi z)}$$

(36.4) Logarithmic derivative of Γ.

Let $\psi(z) := \dfrac{\Gamma'(z)}{\Gamma(z)} : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$. The following properties of $\psi(z)$ follow immediately from their counterparts for Γ.

(1) $$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n}\right)$$

(2) $$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

Note : $$(1) \Rightarrow \psi'(z) = \frac{d^2}{dz^2}\left(\log \Gamma(z)\right) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

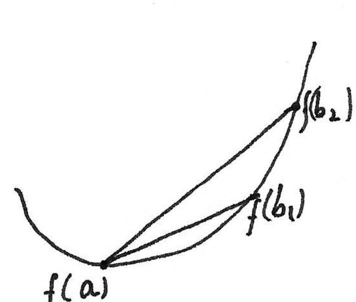
(termwise differentiation is permitted by uniform convergence) i.e. $\psi'(x) > 0 \quad \forall\, x \in \mathbb{R}_{>0}$ - meaning $\log\Gamma : \mathbb{R}_{>0} \to \mathbb{R}$

is <u>a convex function</u>.

<u>Convexity</u> : A function (continuous) $f : \mathbb{R}_{>0} \to \mathbb{R}$ is <u>convex</u>

if $\forall\, a \in \mathbb{R}_{>0}$; $b \mapsto \dfrac{f(b) - f(a)}{b-a}$ is increasing function of $b \in (0,a) \cup (a,\infty)$.

(Note : if f is twice differentiable,
this is equivalent to $f'' > 0$)



Convexity: $b_1 < b_2$ implies

$$\frac{f(b_1) - f(a)}{b_1 - a} \leq \frac{f(b_2) - f(a)}{b_2 - a}$$

(36.5) Bohr- Mollerup Theorem.

Let $F : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuous function such that

(1) $F(x+1) = xF(x) \quad \forall\, x \in \mathbb{R}_{>0}$; (2) $F(1) = 1$.

(3) $f(x) = \log F(x)$ is convex.

Then $F(x) = \Gamma(x) \quad \forall\, x \in \mathbb{R}_{>0}$.

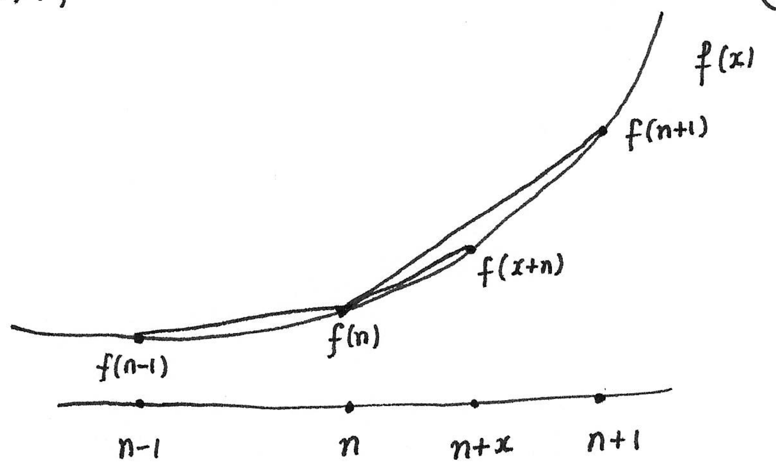
Proof.- Note : by (1) and (2) : $F(n) = (n-1)! \quad \forall\, n \in \mathbb{Z}_{\geq 1}$

and $F(x+n) = (x+n-1) \cdots x\, F(x)$ - i.e. F is completely

determined by its values on $x \in (0,1)$.

Let $n \in \mathbb{Z}_{\geq 2}$ and $x \in (0,1)$

By convexity of $f = \log F$:



$$\frac{f(n-1) - f(n)}{(n-1)-n} \leq \frac{f(x+n) - f(n)}{(x+n) - n}$$

$$\leq \frac{f(n+1) - f(n)}{n+1-n} .$$

increasing slopes - by convexity

Note : $f(n) = \log F(n) = \log (n-1)! = \log 1 + \log 2 + \dots + \log (n-1)$

$\Rightarrow f(n) - f(n-1) = \log (n-1)$; $f(n+1) - f(n) = \log n$.

i.e. $$\log (n-1) \leq \frac{f(x+n) - \log(n-1)!}{x} \leq \log (n)$$

$$\Rightarrow \quad (n-1)! \, (n-1)^x \leq F(x+n) \leq n^x . (n-1)!$$

$$\Rightarrow \quad \frac{(n-1)! \, (n-1)^x}{x(x+1)\dots(x+n-1)} \leq F(x) \leq \frac{(n-1)! \, n^x}{x(x+1)\dots(x+n-1)}$$

As $n \to \infty$; $$\frac{(n-1)!}{x(x+1)\dots(x+n-1)} . (n-1)^x \to \Gamma(x)$$

$\Rightarrow F(x) = \Gamma(x)$

$$\frac{(n-1)!}{x(x+1)\dots(x+n-1)} n^x = \frac{(n-1)! \, (n-1)^x}{x \dots (x+n-1)} \cdot \left(\frac{n}{n-1}\right)^x \to \Gamma(x)$$

$\square$

as $$\lim_{n \to \infty} \left(\frac{n}{n-1}\right)^x = \lim_{n \to \infty} \exp\left(x \cdot \log\left(\frac{n}{n-1}\right)\right) = e^0 = 1.$$