Lecture 37

(37.0) Recall - we proved (Theorem 34.1): Let $G(t,z) : \mathbb{R}_{>0} \times \Omega \to \mathbb{C}$ be a continuous function such that, for a fixed $t_0 \in \mathbb{R}_{>0}$, the resulting function $G(t_0, z) : \Omega \to \mathbb{C}$ is holomorphic. Assume that $\lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R G(t,z)\,dt$ converges uniformly relative to compact sets in Ω.

Then $g(z) := \int_0^\infty G(t,z)\,dt \; : \Omega \to \mathbb{C}$ is holomorphic and

$$g'(z) = \int_0^\infty G_z(t,z)\,dt.$$

We used it to define $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$. $(Re(z) > 0)$.

Main Properties: $\Gamma(z+1) = z\Gamma(z)$; $\Gamma(1) = 1$; $\Gamma(n) = (n-1)!$.

- Γ extends to a mero. fn. $\mathbb{C} \dashrightarrow \mathbb{C}$. It has poles at $\mathbb{Z}_{\leq 0}$ - all simple and $\operatorname{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}$. $\Gamma'(1) = -\gamma$ (γ = Euler-Mascheroni constant)

- $$\Gamma(z) = \lim_{N\to\infty} N^z \frac{N!}{z(z+1)\cdots(z+N)} = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

- $$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

- $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ satisfies: (i) $\psi(z+1) - \psi(z) = \dfrac{1}{z}$

(ii) $\psi(1) = -\gamma$ (iii) $\psi(z) = \lim\limits_{N\to\infty} \left(\ln(N) - \sum\limits_{k=0}^{N} \dfrac{1}{z+k} \right)$

$$= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right)$$

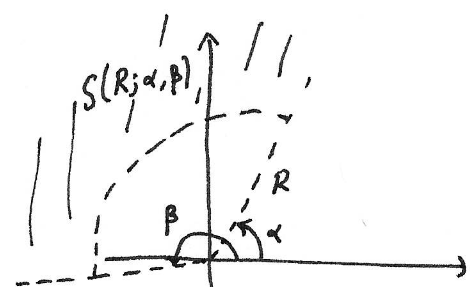
$\psi'(z) = \sum\limits_{k=0}^{\infty} \dfrac{1}{(z+k)^2}$ solves $\psi'(z+1) - \psi'(z) = -\dfrac{1}{z^2}$.

(37.1) Stirling Series for $\Gamma(z)$ is its asymptotic expansion as $z\to\infty$, $Re(z)>0$.

Definition. Let $\alpha < \beta$; ~~$\alpha, \beta \in (-\pi, \pi]$~~ $\beta - \alpha \in (0, 2\pi)$.

$$S(R;\alpha,\beta) = \{ z \in \mathbb{C} : |z| > R,\ \alpha < \arg(z) < \beta \}$$

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that there exist R, α, β s.t. $S(R;\alpha,\beta) \subset \Omega$ (domain of f).



Let $\sum\limits_{n=0}^{\infty} c_n z^{-n}$ be a power series in z^{-1}; $c_0, c_1, \ldots \in \mathbb{C}$.

We say $f(z) \sim \sum\limits_{n=0}^{\infty} c_n z^{-n}$ as $z \to \infty$, $z \in S(R;\alpha,\beta)$

if for every $N \geq 0$, $\lim\limits_{\substack{z\to\infty \\ z \in S(R;\alpha,\beta)}} \left(f(z) - \sum\limits_{n=0}^{N-1} c_n z^{-n} \right) \cdot z^N$ exists.

Slightly more generally, by $f \sim g \cdot (1 + c_1 z^{-1} + \dots)$ as $z \to \infty$, $\alpha < \arg(z) < \beta$

we mean $$\frac{f(z)}{g(z)} \sim 1 + c_1 z^{-1} + \dots$$

Remarks. Asymptotic expansions behave exactly in the same manner as Taylor series expansion.

e.g. $$f \sim \sum_{n=0}^{\infty} c_n z^{-n}, \quad g \sim \sum_{n=0}^{\infty} d_n z^{-n} \quad \Rightarrow \quad \alpha f + \beta g \sim \sum_{n=0}^{\infty} (\alpha c_n + \beta d_n) z^{-n}$$

$$f \cdot g \sim \sum_{N=0}^{\infty} z^{-N} (c_N d_0 + \dots + c_0 d_N)$$

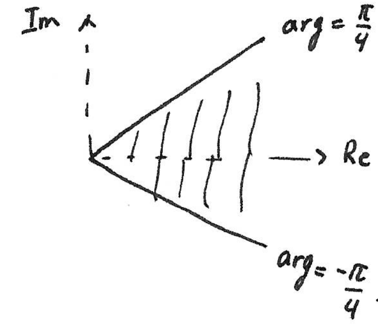
For sectors of <u>non-zero opening</u>, termwise differentiation / integration is permissible.

$$f \sim \sum_{n=0}^{\infty} c_n z^{-n} \quad \Rightarrow \quad f' \sim \sum_{n=0}^{\infty} -n c_n z^{-n-1}$$

$$F'(z) = f \quad \Rightarrow \quad F \sim c_0 z + c_1 \log(z) + \sum_{n=2}^{\infty} c_n \frac{z^{-n+1}}{-n+1} + \text{constant}$$

One major difference is that <u>different functions can have the same asymptotic expansion</u>. e.g. $e^{-z} \sim 0$ in sector $S(0; \frac{-\pi}{4}, \frac{\pi}{4})$

since $$\lim_{\substack{\cancel{z \to \infty} \\ \cancel{Re(z) > 0} \\ Re(z) \to \infty}} z^N e^{-z} = 0 \quad \forall N \geq 0.$$



(37.2) Laplace transform and asymptotic expansions. [Optional]

The definition of asymptotic expansion is due to Poincaré, and as stated here, is not the most general one. [Suggested reading: Chapter 6 of Ablowitz-Fokas - Complex Variables].

Historically, the motivation for studying asymptotic expansions was two-fold.

(1) Solving difference-differential equations

(2) Finding "sum" of a divergent series. [see "Divergent Series" by G. H. Hardy].

Example (1). Consider the equation. $F(x+1) = F(x) - \frac{1}{x^2}$.

We can formally solve it: let $F(x) = \sum_{n=0}^{\infty} c_n x^{-n-1}$. Note:

$$(x+1)^{-n-1} = x^{-n-1} \cdot (1+x^{-1})^{-n-1} = x^{-n-1} \sum_{l=0}^{\infty} (-1)^l \binom{n+l}{l} x^{-l}.$$

Hence,

$$F(x+1) = \sum_{n=0}^{\infty} c_n (x+1)^{-n-1} = \sum_{n=0}^{\infty} c_n x^{-n-1} \left(\sum_{l=0}^{\infty} (-1)^l \binom{n+l}{l} x^{-l} \right)$$

$$= \sum_{N=0}^{\infty} x^{-N-1} \left(\sum_{l=0}^{N} (-1)^l \binom{N}{l} c_{N-l} \right).$$

$$F(x+1) - F(x) = \sum_{N=0}^{\infty} x^{-N-1} \left(\sum_{l=1}^{N} (-1)^l \binom{N}{l} c_{N-l} \right) = -x^{-2}.$$

e.g. coeff. of x^{-2}: $-c_0 = -1$ $\Rightarrow c_0 = 1$; coeff. of x^{-3}: $-2c_1 + c_0 = 0$ $\Rightarrow c_1 = \frac{1}{2}$ and so on...

The resulting series $\sum_{n=0}^{\infty} c_n x^{-n-1}$ turns out to be divergent - - and in this case - it is the asymptotic expansion of

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} \quad \text{as } z \to \infty, \ Re(z) > 0.$$

(2): given $a_0, a_1, a_2, \dots \in \mathbb{C}$, we define $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} (a_0 + \dots + a_N)$ and call the series <u>convergent</u> or <u>divergent</u> - according to whether the limit exists or not (Cauchy). However, there could be a method of assigning numerical value to $a_0 + a_1 + \dots$; even when $\sum_{k=0}^{\infty} a_k$ is divergent.

Example. - <u>Cesàro sum</u> - given $\{a_n\}_{n=0}^{\infty}$, let $S_N = a_0 + \dots + a_N$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be Cesàro summable, if

$$\lim_{N\to\infty} \frac{S_0 + \dots + S_N}{N+1} \quad \text{exists} \quad \left(\sum_{n=0}^{\infty} a_n \stackrel{C}{=} \lim_{N\to\infty} \frac{S_0 + \dots + S_N}{N+1} \right).$$

<u>Abel summation</u> - given $\{a_n\}_{n=0}^{\infty}$ s.t. $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $= 1$.

We say $\sum_{n=0}^{\infty} a_n$ is Abel-summable if $\lim_{\substack{r\to 1 \\ 0<r<1}} \sum_{n=0}^{\infty} a_n r^n$ exists.

$$\left(\sum_{n=0}^{\infty} a_n \stackrel{A}{=} \lim_{\substack{r\to 1 \\ (0<r<1)}} \sum_{n=0}^{\infty} a_n r^n \right)$$

Example: $a_n = (-1)^n$. $\sum_{n=0}^{\infty} (-1)^n$ has partial sums $1, 0, 1, 0, \dots$ hence is divergent.

But $\sum_{n=0}^{\infty} (-1)^n \overset{C}{=} \frac{1}{2}$ $\Big(S_0 = 1,\ S_1 = 0,\ S_2 = 1,\ \dots$

$$\frac{S_0 + \dots + S_N}{N+1} = \begin{cases} \frac{1}{2} & \text{if } N \text{ is odd} \\ \frac{1}{2} + \frac{1}{N+1} & \text{if } N \text{ is even} \end{cases}$$

$$\to \frac{1}{2} \text{ as } N \to \infty \Big)$$

and $$\sum_{n=0}^{\infty} (-1)^n \overset{A}{=} \lim_{\substack{r \to 1 \\ 0<r<1}} \sum_{n=0}^{\infty} (-1)^n r^n = \lim_{r \to 1} \frac{1}{1+r} = \frac{1}{2}.$$

A method of summation, due to Borel, uses Laplace transform - very useful integral transform in these problems.

(37.3) Laplace transform. Let $\phi(t)$ be a function of $t \in \mathbb{R}_{>0}$.

$$\mathcal{L}\phi(z) := \int_0^{\infty} \phi(t) e^{-zt}\, dt \quad (\text{Laplace transform of } \phi).$$

Lemma. Assume $\phi(t)$ satisfies the following 2 conditions.

(1) $\phi(t)$ has at most exponential growth as $t \to \infty$. Meaning:
$\exists\, R > 0,\ M, C > 0$ s.t. $|\phi(t)| < M \cdot e^{ct}$ $\forall\, t > R$.

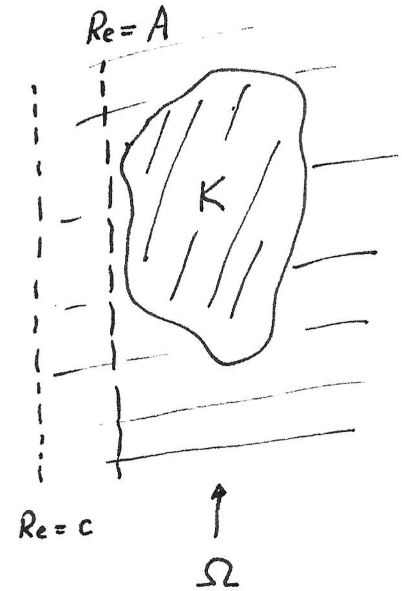
(2) $\phi(t)$ has at most logarithmic singularity as $t \to 0$.
Meaning: $\exists\, r > 0;\ a; b > 0$ s.t. $|\phi(t)| < b \cdot t^{a-1}$ $\forall\, t < r$.

Then $\int_0^{\infty} \phi(t) e^{-zt}\, dt$ converges uniformly (rel. to compact sets in $\Omega = \{\mathrm{Re}(z) > c\}$.

Proof. - We need to verify that given any compact set K in $\Omega = \{Re(z) > c\}$ and $\varepsilon > 0$; we can find $r_1 > 0$, $R_1 > 0$ s.t.

$$\left| \int_0^s \phi(t) e^{-zt} dt \right| < \varepsilon \qquad \forall\ 0 < s < r_1,\ z \in K.$$

$$\left| \int_S^\infty \phi(t) e^{-zt} dt \right| < \varepsilon \qquad \forall\ R_1 < S,\ z \in K.$$



Near ∞; Let $A > c$ be s.t. $A < Re(z)$ $\forall z \in K$.

choose $\underset{R_1}{\cancel{S}} > R$ s.t. $e^{-(A-c)R_1} < \dfrac{\varepsilon(A-c)}{M}$.

Then $\forall\ S > R_1$ we have

$$\left| \int_S^\infty \phi(t) e^{-zt} dt \right| \leq \int_S^\infty |\phi(t)| \cdot e^{-Re(z)t} dt$$

$$< M \int_S^\infty e^{-(A-c)t} dt = M \cdot \frac{e^{-(A-c)S}}{A-c} < \varepsilon.$$

Near 0; choose $r_1 < r$ s.t. $r_1^a < \varepsilon \cdot \dfrac{a}{b}$. Then $\forall\ s < r_1$ we have

$$\left| \int_0^s \phi(t) e^{-zt} dt \right| < \int_0^s b \cdot t^{a-1} dt = b \frac{s^a}{a} < \varepsilon.$$

$\square$

(37.4) Examples.

(i) $\phi(t) = \dfrac{t^n}{n!}$ $(n \in \mathbb{Z}_{\geq 0})$ $\Rightarrow$ $\mathcal{L}\phi(z) = \dfrac{1}{z^{n+1}}$ $(Re(z)>0)$.

Proof - by induction on n. Let $p_n(t) = \dfrac{t^n}{n!}$.

$(n=0)$
$$\mathcal{L}p_0(z) = \int_0^\infty e^{-zt}\,dt = \left[\frac{e^{-zt}}{-z}\right]_0^\infty$$

$$= \underbrace{\cancel{\lim_{t\to\infty} \frac{e^{-zt}}{-z}}}_{0 \text{ if } Re(z)>0} + \frac{1}{z} = \frac{1}{z}.$$

Induction step.
$$\mathcal{L}p_{n+1}(z) = \int_0^\infty \frac{t^{n+1}}{(n+1)!} e^{-zt}\,dt$$

$$= \underbrace{\cancel{\left[\frac{t^{n+1}}{(n+1)!}\,\frac{e^{-zt}}{(-z)}\right]_0^\infty}}_{0 \ (Re(z)>0)} + \frac{1}{z}\int_0^\infty \frac{t^n}{n!} e^{-zt}\,dt \quad \left(\begin{array}{l}\text{integration}\\ \text{by parts}\end{array}\right)$$

$$= \frac{1}{z}\cdot\frac{1}{z^{n+1}} \quad \text{(by induction)}$$

$$= \frac{1}{z^{n+2}}.$$

(ii) $\phi(t) = e^{\kappa t}$ $(\kappa \in \mathbb{R})$ $\Rightarrow$ $\mathcal{L}\phi(z) = \dfrac{1}{z-\kappa}$ for $Re(z) > \kappa$.

$$\int_0^\infty e^{\kappa t} e^{-zt}\,dt = \left[\frac{e^{(\kappa - z)t}}{\kappa - z}\right]_0^\infty = \frac{1}{z-\kappa} \quad \text{if } Re(z) > \kappa.$$