Lecture 38

(38.0) Recall - last time we defined the notion of asymptotic (power series) expansion of a function:

$$f(z) \sim \sum_{n=0}^{\infty} c_n z^{-n} \quad \text{as } z \to \infty,\ z \in S$$

means:

$$\lim_{\substack{z\to\infty \\ z\in S}} z^N \left(f(z) - \sum_{n=0}^{N-1} c_n z^{-n} \right) \text{ exists } \forall\, N \geq 0. \qquad -(*)$$

Here S is some sector near ∞ (i.e. $S = \{z \in \mathbb{C} \mid |z| > R,\ \phi_1 < \arg(z) < \phi_2\}$ for some R, ϕ_1, ϕ_2.)

Remarks - (i) More general notion of asymptotics exist. See, e.g.,

[O. Costin: Asymptotics and Borel summability.
Ablowitz - Fokas: Complex Variables (Chapter 6).]

e.g. one could start with an arbitrary sequence of functions $\{\delta_n(z)\}_{n=0}^{\infty}$ such that $\dfrac{\delta_{n+1}(z)}{\delta_n(z)} \to 0$ as $z \to \infty$, $z \in S$.

and consider asymptotic expansions of the form $\sum_{n=0}^{\infty} a_n \delta_n(z)$. We have narrowed our attention to $\delta_n(z) = z^{-n}$ - power series case.

(ii) $(*)$ in the definition of $f(z) \sim \sum_{n=0}^{\infty} c_n z^{-n}$ can alternately be written as:

$$\lim_{\substack{z\to\infty \\ z\in S}} z^{N-1} \left(f(z) - \sum_{n=0}^{N-1} c_n z^{-n} \right) = 0 \quad \forall\, N \geq 1.$$

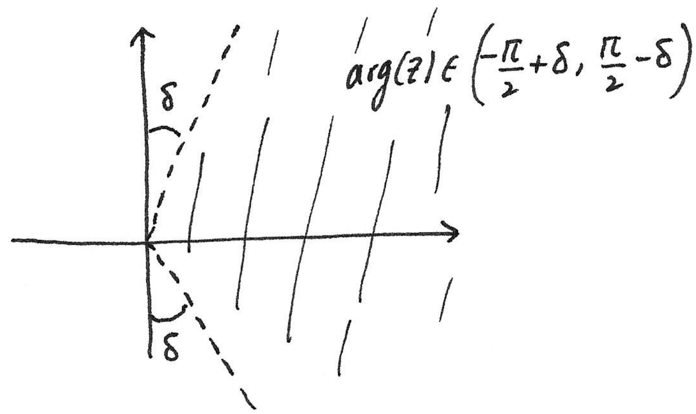
(38.1) Watson's Lemma. - Let $\phi : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function having at most exponential growth as $t \to \infty$: (See Lemma 37.3)

Meaning $\exists\, R, M, C > 0$ s.t. $|\phi(t)| < M \cdot e^{Ct}$ $\forall\, t > R$.

Assume that $\phi(t) \sim \sum_{n=0}^{\infty} a_n t^n$ as $t \to 0^+$ (i.e. $t \in \mathbb{R}_{>0}$, $t \to 0$).

Lemma. $$\mathcal{L}\phi(z) = \int_0^{\infty} \phi(t)\, e^{-zt}\, dt \sim \sum_{n=0}^{\infty} a_n \cdot \frac{n!}{z^{n+1}}$$

as $z \to \infty$; $\arg(z) \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$ ($\delta > 0$ any positive real)



Proof.

Note : for $\arg(z) \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right)$

$$|z| \cdot \sin(\delta) \leq \mathrm{Re}(z) \leq |z|$$

Therefore, $z \to \infty$ in this sector $\iff$ $\mathrm{Re}(z) = x \to \infty$.

Let $N \geq 1$. We have to prove that :

$$\lim_{\substack{z \to \infty \\ \arg(z) \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right)}} z^N \left(f(z) - \sum_{n=0}^{N-1} \frac{n!\, a_n}{z^{n+1}} \right) = 0 \quad ; \quad f(z) = \mathcal{L}\phi(z) = \int_0^{\infty} \phi(t)\, e^{-zt}\, dt$$

(or $x = \mathrm{Re}(z) \to \infty$)

Note : $$f(z) - \sum_{n=0}^{N-1} \frac{a_n \cdot n!}{z^{n+1}} = \int_0^{\infty} \left(\phi(t) - \sum_{n=0}^{N-1} a_n t^n\right) e^{-zt}\, dt$$

(since Laplace transform of $\frac{t^n}{n!}$ is z^{-n-1} - see example (37.4)(i).)

Let $$\phi_N(t) = \phi(t) - \sum_{n=0}^{N-1} a_n t^n \sim a_N t^N + a_{N+1} t^{N+1} + \cdots$$ as $t \to 0^+$.

i.e. $$\lim_{t \to 0^+} \frac{\phi_N(t)}{t^N} = a_N \text{ exists.}$$

$\Rightarrow$ we can find $r > 0$ and $A > 0$ s.t. $|\phi_N(t)| < A \cdot t^N$ for $t \in (0, r)$. [Check - why?]

Let us write $$\int_0^{\infty} \phi_N(t) e^{-zt}\, dt = \int_0^{r} \phi_N(t) e^{-zt}\, dt + \int_r^{R} \phi_N(t) e^{-zt}\, dt + \int_R^{\infty} \phi_N(t) e^{-zt}\, dt$$

where $R > 0$ is such that $\exists\, M, C > 0$ with $|\phi_N(t)| < M \cdot e^{ct}$, $\forall\, t > R$.

(exists since ϕ has at most exponential growth as $t \to \infty$, $\phi_N = \phi -$ a polynomial in t and polynomial functions also have at most exponential growth at ∞).

$$\int_0^\infty \phi_N(t)\, e^{-zt}\, dt = \underbrace{\int_0^r \phi_N(t)\, e^{-zt}\, dt}_{①} + \underbrace{\int_r^R \phi_N(t)\, e^{-zt}\, dt}_{②} + \underbrace{\int_R^\infty \phi_N(t)\, e^{-zt}\, dt}_{③}$$

We will see that ② and ③ are asymptotically zero.

② : Since ϕ_N is continuous, let $m = \mathrm{Max}\{|\phi_N(t)| : t \in [r, R]\}$.

Then $\left| \int_r^R \phi_N(t)\, e^{-zt}\, dt \right| \leq m \cdot \int_r^R e^{-xt}\, dt$ $\quad (x = \mathrm{Re}(z))$

$$= m \cdot \frac{e^{-rx} - e^{-Rx}}{x} \to 0 \text{ as } x \to \infty$$

faster than $x^N \to \infty$

i.e. $\lim\limits_{\substack{z \to \infty \\ \arg(z) \in \left(-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\right)}} z^N \cdot \int_r^R \phi_N(t)\, e^{-zt}\, dt \rightrightarrows 0$.

③ : $\left| \int_R^\infty \phi_N(t)\, e^{-zt}\, dt \right| \leq M \int_R^\infty e^{(c-x)t}\, dt = \dfrac{M \cdot e^{-(x-c)R}}{x-c} \to 0$ as $x \to \infty$. exponentially.

Finally, $\left| \int_0^r \phi_N(t)\, e^{-zt}\, dt \right| \leq A \int_0^r t^N e^{-xt}\, dt < A \int_0^\infty t^N e^{-xt}\, dt$

$$= A \cdot \frac{N!}{x^{N+1}}$$

$$\Rightarrow \left| z^N \int_0^r \phi_N(t)\, e^{-zt}\, dt \right| \leq A \cdot N! \cdot \frac{|z|^N}{Re(z)^{N+1}}$$

$$\leq \frac{A \cdot N!}{\sin(\delta)^N} \frac{1}{Re(z)} \to 0 \quad \text{as } Re(z) \to \infty.$$

(recall: $|z| \sin\delta \leq Re(z) \leq |z|$). □

(38.2) Applications of Watson's Lemma.

Recall: $\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(\left(1+\frac{z}{n}\right)^{-1} e^{z/n} \right)$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right)$$

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}. \quad \text{Note: } \psi'(z+1) = \psi'(z) - \frac{1}{z^2}.$$

Solving difference equation $F(z+1) - F(z) = \frac{-1}{z^2}$ using Laplace transform.

Let $F(z) = \int_0^{\infty} \phi(t)\, e^{-zt}\, dt$. Then

$$F(z) - F(z+1) = \int_0^{\infty} (1-e^{-t})\, \phi(t)\, e^{-zt}\, dt = \frac{1}{z^2} = \int_0^{\infty} t \cdot e^{-zt}\, dt$$

$\Rightarrow$ for $\phi(t) = \frac{t}{1-e^{-t}}$; $F(z) = \int_0^{\infty} \frac{t}{1-e^{-t}} e^{-zt}\, dt$ solves

$$F(z+1) = F(z) - \frac{1}{z^2}.$$

Alternately, $$\sum_{K=0}^{\infty} \frac{1}{(z+K)^2} = \sum_{K=0}^{\infty} \int_0^{\infty} t \cdot e^{-(z+K)t}\, dt \qquad (\mathrm{Re}(z)>0)$$

and $$\sum_{K=0}^{\infty} e^{-Kt} = \frac{1}{1-e^{-t}}$$ uniformly in $t \in \mathbb{R}_{>0}$.

$$\Rightarrow \psi'(z) = \sum_{K=0}^{\infty} \frac{1}{(z+K)^2} = \int_0^{\infty} \frac{t}{1-e^{-t}}\, e^{-zt}\, dt.$$

By Watson's Lemma: $$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \sum_{K=1}^{\infty} \frac{B_{2K}}{z^{2K+1}} \quad \text{as } z \to \infty$$

$$\left(\arg(z) \in \left(\tfrac{-\pi}{2}+\delta, \tfrac{\pi}{2}-\delta\right) \right)$$

$$\left(\text{recall: } \frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=0}^{\infty} B_{2K} \frac{t^{2K}}{(2K)!} \right)$$

(38.3) $$\psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{K=1}^{\infty} \frac{B_{2K}}{2K} z^{-2K} \qquad \left(\begin{array}{l} \text{as } z \to \infty \\ \arg(z) \in \left(\tfrac{-\pi}{2}+\delta, \tfrac{\pi}{2}-\delta\right) \end{array} \right)$$

Write $$\psi'(z) = \int_0^{\infty} \left[1 + \left(\frac{t}{1-e^{-t}} - 1 \right) \right] e^{-zt}\, dt$$

$$= \frac{1}{z} + \int_0^{\infty} \left(\frac{t}{1-e^{-t}} - 1 \right) e^{-zt}\, dt$$

$$\Rightarrow \psi(z) = C + \log(z) + \int_0^{\infty} \left(\frac{t}{1-e^{-t}} - 1 \right) \frac{e^{-zt}}{(-t)}\, dt$$

$$= C + \log(z) - \int_0^{\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-zt}\, dt$$

Recall : $\psi(1) = -\gamma$. The following claim implies (upon setting $z = 1$) that the constant of integration, $C = 0$.

<u>Claim 1.</u> $$\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t}\, dt = \gamma.$$ (Proof given in (38.5) below)

~~38.4~~ Using $$\frac{1}{1-e^{-t}} - \frac{1}{t} = \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k-1}}{(2k)!}$$ and using

Watson's Lemma again implies:
$$\psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} z^{-2k} \quad \text{as} \quad z \to \infty \quad \left(\arg(z) \in \left(-\tfrac{\pi}{2}+\delta, \tfrac{\pi}{2}-\delta\right)\right).$$

(38.4) $$\log \Gamma(z) \sim \left(z - \frac{1}{2}\right) \log(z) - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$

(Stirling series for gamma function).

Again, starting from $$\psi(z) = \log(z) - \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} + \frac{1}{2} - \frac{1}{2} \right) e^{-zt}\, dt$$
$$= \log(z) - \frac{1}{2z} - \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right) e^{-zt}\, dt$$

We get
$$\log \Gamma(z) = z \log(z) - z - \frac{1}{2}\log(z) + \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right) \frac{e^{-zt}}{t}\, dt$$
$$+ D$$ (D: constant of integration)

Claim 2. $$\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)\frac{e^{-t/2}}{t}\,dt = \frac{1}{2}\left(1 - \ln(2)\right)$$

Assuming this, and setting $z = \frac{1}{2}$ in the expression for $\log\Gamma(z)$ on the previous page (recall $\Gamma(\frac{1}{2}) = \sqrt{\pi}$), we get

$$\frac{1}{2}\ln(\pi) = -\cancel{\tfrac{1}{2}} + D + \cancel{\tfrac{1}{2}} - \frac{1}{2}\ln(2) \implies D = \frac{1}{2}\ln(2\pi)$$

Hence we get:

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log(z) - z + \ln\left(\sqrt{2\pi}\right) + \int_0^\infty \underbrace{\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)\frac{e^{-tz}}{t}}_{\sum_{k=1}^\infty \frac{B_{2k}\, t^{2k-2}}{(2k)!}}\,dt$$

(for $Re(z) > 0$)

$$\sim \left(z - \frac{1}{2}\right)\log(z) - z + \ln\left(\sqrt{2\pi}\right) + \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)} z^{-2k+1} \quad \text{as } z \to \infty,\ -\frac{\pi}{2} + \delta < \arg(z) < \frac{\pi}{2} - \delta$$

(38.5) Some real integrals and proofs of claims 1 (p.7) and 2.

1. $$\log(z) = \int_0^\infty \frac{e^{-t} - e^{-tz}}{t}\,dt. \quad \forall z \text{ in } \{\text{Real part} > 0\}.$$

Proof. Note that $G(t,z) = \dfrac{e^{-t} - e^{-tz}}{t}$ satisfies conditions of Theorem 34.1, for $(t,z) \in \mathbb{R}_{>0} \times \Omega$ ($\Omega = \{z \mid Re(z) > 0\}$).

Hence $F(z) = \displaystyle\int_0^\infty \frac{e^{-t} - e^{-tz}}{t}\,dt$ is a holomorphic function of $z \in \Omega$. Note: $F(1) = 0$ & $F'(z) = \displaystyle\int_0^\infty e^{-tz}\,dt = \frac{1}{z}$.

$$\Rightarrow \quad F(z) = \log(z).$$

2. <u>Proof of claim 1.</u> $\gamma = 1 + \frac{1}{2} + \dots + \frac{1}{N} - \ln(N) \quad \left(\lim_{N\to\infty}\right)$

$$= \lim_{N\to\infty} \left(\sum_{k=1}^{N} \int_0^\infty e^{-kt}\, dt - \int_0^\infty \frac{e^{-t} - e^{-Nt}}{t}\, dt \right)$$

$$= \lim_{N\to\infty} \int_0^\infty \left(\frac{e^{-t} - e^{-Nt}}{1 - e^{-t}} - \frac{e^{-t} - e^{-Nt}}{t} \right) dt$$

$$= \int_0^\infty \left(\frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right) dt .$$

3. <u>Proof of claim 2.</u> Recall $B(t) := \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!} = \frac{t}{e^t - 1} - 1 + \frac{t}{2}$

$$= \frac{t}{1 - e^{-t}} - 1 - \frac{t}{2}.$$

Let $$J = \int_0^\infty \frac{B(t)}{t^2} e^{-t/2}\, dt$$

$$I = \int_0^\infty \frac{B(u)}{u^2} e^{-u}\, du = \int_0^\infty \frac{2\,B(t/2)}{t^2} e^{-t/2}\, dt \quad (\text{set } u = t/2)$$

$$\Rightarrow \quad J = (J - I) + I$$

$$= \int_0^\infty \left(\frac{B(t)}{t^2} - \frac{2\,B(t/2)}{t^2} \right) e^{-t/2}\, dt + \int_0^\infty \frac{B(t)}{t^2} e^{-t}\, dt$$

Integrand of $J - I$: $\left(B(t) - 2B(t/2)\right) \dfrac{e^{-t/2}}{t^2}$

$$= \left(\frac{t}{e^t - 1} - 1 + \cancel{\frac{t}{2}} - \cancel{2} \frac{t/\cancel{2}}{e^{t/2} - 1} + 2 - 2 \cdot \cancel{\frac{t}{4}} \right) \frac{e^{-t/2}}{t^2}$$

$$= \left(1 - \frac{t e^{t/2}}{e^t - 1} \right) \frac{e^{-t/2}}{t^2} = \frac{e^{-t/2}}{t^2} - \frac{1}{t(e^t - 1)}$$

$$\Rightarrow \text{ Integrand of } J = \frac{e^{-t/2}}{t^2} - \frac{\cancel{1}}{t(\cancel{e}^t - 1)} + \frac{e^{-t}}{t^2}\left(\frac{t\cancel{\ }}{1 \cancel{-} e^{-t}} - 1 - \frac{t}{2} \right)$$

$$= -\frac{d}{dt}\left(\frac{e^{-t/2} - e^{-t}}{t} \right) + \frac{1}{2} \frac{e^{-t} - e^{-t/2}}{t}.$$

$$\Rightarrow \quad J = -\left[\frac{e^{-t/2} - e^{-t}}{t} \right]_0^\infty + \frac{1}{2} \int_0^\infty \frac{e^{-t} - e^{-\frac{1}{2}t}}{t}\, dt$$

$$= \frac{1}{2} + \frac{1}{2} \ln\left(\frac{1}{2}\right).$$