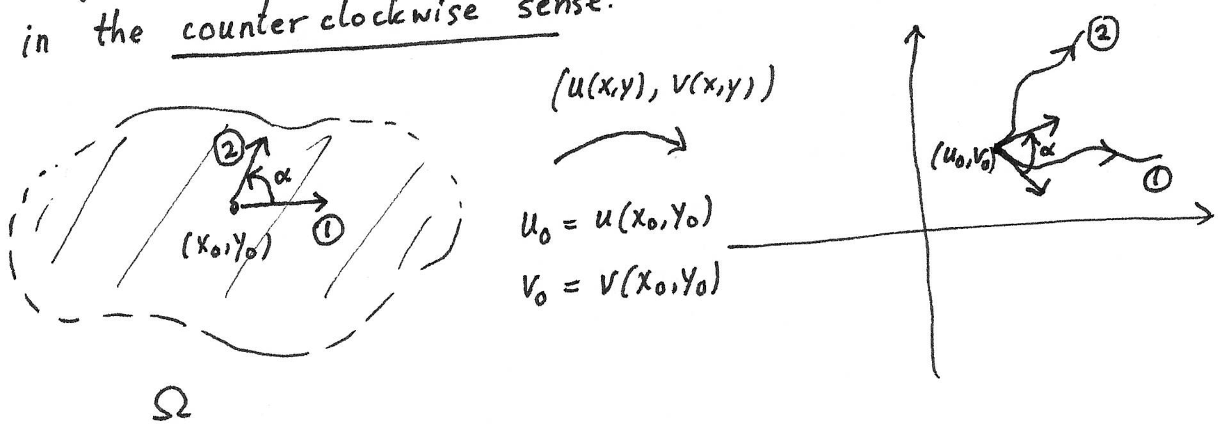
Lecture 39

Geometric properties of holomorphic functions.

(39.0) The geometric significance of holomorphic functions lies in their "angle preservation". Historically speaking, the problem of drawing a "faithful map" of earth on a 2-dimensional sheet of paper has attracted the attention of scholars since ancient times. "Faithful" here means a drawing that respects angles incident at a point. Most notably - stereographic projection was discovered by Ptolemy in 150 CE and used by Gerard Mercator in 1569 – in order to assist navigation.

A <u>conformal map</u> is a function $\Omega \to \mathbb{R}^2$ ($\Omega \subset \mathbb{R}^2$ open, connected) which preserves angles incident at each point $(x_0, y_0) \in \Omega$.

Remark. - Angles between two vectors are assumed to be measured in the <u>counter clockwise sense</u>.

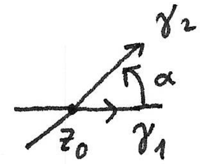


(39.1) Let $\Omega \subset \mathbb{C}$ be an open, connected set. $f : \Omega \to \mathbb{C}$ be a holomorphic function.

Lemma. Let $z_0 \in \Omega$ be such that $f'(z_0) \neq 0$. Then f preserves angles incident at z_0.

In more detail, consider $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \Omega$

$$\gamma_1(t) = z_0 + t$$
$$\gamma_2(t) = z_0 + t e^{i\alpha}$$



(assume $0 \leq \alpha \leq \pi$ for definiteness)

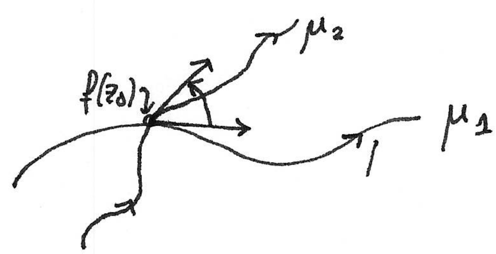
($\varepsilon > 0$ is small enough so that $D(z_0; \varepsilon) \subset \Omega$).

Let $\mu_1(t) = f(z_0 + t)$, $\mu_2(t) = f(z_0 + t e^{i\alpha})$ $(t \in (-\varepsilon, \varepsilon))$ two curves passing through $f(z_0)$ at $t = 0$.

Then the angle between the tangent vectors $\mu_1'(0)$ and $\mu_2'(0)$ is α.

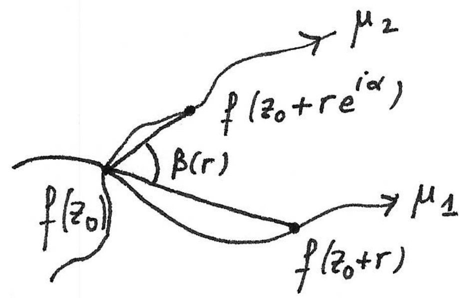
Proof.-

Let $0 < r < \varepsilon$ and consider points $f(z_0 + r)$ and $f(z_0 + r \cdot e^{i\alpha})$ lying on μ_1 and μ_2 respectively.



$$\beta(r) = \arg \left\{ \frac{f(z_0 + r e^{i\alpha}) - f(z_0)}{f(z_0 + r) - f(z_0)} \right\}$$

= angle between line segments joining $f(z_0)$ to $f(z_0 + r e^{i\alpha})$ and $f(z_0 + r)$ respectively.



$$\beta = \lim_{r \to 0} \beta(r) = \text{angle between the tangent vectors } \mu_1'(0) \text{ and } \mu_2'(0)$$

Note:
$$f(z_0 + r) - f(z_0) = r f'(z_0) + r^2(\cdots)$$
$$f(z_0 + r e^{i\alpha}) - f(z_0) = r \cdot e^{i\alpha} f'(z_0) + r^2(\cdots)$$

$$\Rightarrow \quad \beta = \lim_{r \to 0} \arg \left\{ \frac{e^{i\alpha} + r(\cdots)}{1 + r(\cdots)} \right\} = \arg\left(e^{i\alpha}\right)$$

$$= \alpha \qquad \square$$

(39.2) Remark.- If $f'(z_0) = 0$, the angles incident at z_0 get magnified by m ; where $m = \text{smallest}\{k \text{ s.t. } f^{(k)}(z_0) \neq 0\}$.

e.g. let $f(z) = z^m$ $(m \in \mathbb{Z}_{\geq 0})$ and $z_0 = 0$. Then

$$f(re^{i\alpha}) = r^m e^{im\alpha}$$
$$f(r) = r^m$$

$\rightsquigarrow$ angle between $f(r)$ $(r \in \mathbb{R})$ and $f(re^{i\alpha})$

$= m\alpha$ (modulo 2π).

(39.3) Area of a domain under a holomorphic function

Lemma. Let $f : \Omega \to \mathbb{C}$ be a one to one, holomorphic function. Then

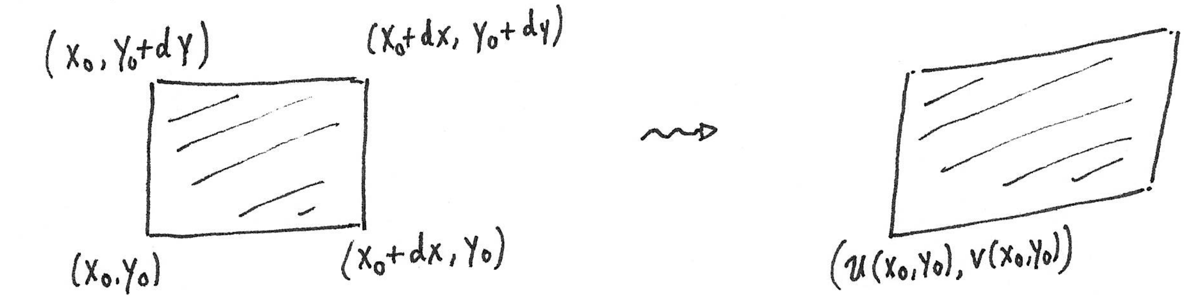
$$\text{Area}(f(\Omega)) = \iint_{\Omega} |f'(z)|^2 \, dx\, dy$$

(in literature the word <u>univalent</u> is also used to mean one-one)

Proof. $\text{Area}(f(\Omega)) = \iint_{f(\Omega)} dx\, dy$ by definition. If $f = u + iv$,

then the change of variables $x \rightsquigarrow u(x,y)$, $y \rightsquigarrow v(x,y)$ (re parametrization of $f(\Omega)$ - since f is univalent) changes the area element $dxdy$ to

$\left| u_x v_y - u_y v_x \right| dx\, dy$. To see this, we draw a small rectangle of sides dx, dy and compute the area of the parallelogram:



The sides of the parallelogram are given by vectors $dx \langle u_x, v_x \rangle$, $dy \langle u_y, v_y \rangle$ and its area is the magnitude of the cross product

$$\left| u_x v_y - u_y v_x \right| \cdot dx \cdot dy .$$

Since f is holomorphic, $u_x = v_y$, $u_y = -v_x$ (Cauchy - Riemann equations)

and hence $\left| u_x v_y - u_y v_x \right| = \left| f'(z) \right|^2 = u_x^2 + u_y^2$ $\square$

Remark. - If the assumption of f being 1-1 is dropped, the expression $\iint_\Omega \left| f'(z) \right|^2 dx\, dy$ will compute the area of $f(\Omega)$ with "multiplicity"

e.g. $f(z) = z^2$; $\Omega = D(0;1)$; $\left| f'(z) \right|^2 = 4 \cdot |z|^2$

$$\iint_{D(0;1)} \left| f'(z) \right|^2 dx\, dy = 4 \iint_{D(0;1)} (x^2 + y^2)\, dx\, dy = 4 \int_0^{2\pi} \int_0^1 r^2 \cdot r \cdot dr \cdot d\theta$$

(polar coordinates)

$$= 4 \cdot \left[\frac{r^4}{4} \right]_0^1 \left[\theta \right]_0^{2\pi} = 2\pi = 2 \cdot \text{Area}\left(D(0;1) \right)$$

Since $f = z^2 : D(0;1) \to D(0;1)$ is $2:1$ map.

(39.4) Example. - Holomorphic functions given by power series.

Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n : D(0;R) \to \mathbb{C}$ $(R>0)$

is $1:1$ (or univalent). Then $f'(z) = \sum_{n=1}^{\infty} n \cdot c_n \cdot z^{n-1}$ and

$$|f'(z)|^2 = f'(z) \cdot \overline{f'(z)} = \left(\sum_{n=1}^{\infty} n\, c_n z^{n-1}\right)\left(\sum_{m=1}^{\infty} m\, \overline{c_m}\, (\overline{z})^{m-1}\right)$$

$$= \sum_{n,m\geq 1} n\, m\, c_n \overline{c_m}\, z^{n-1} (\overline{z})^{m-1}.$$

$$\text{Area}(f(D(0;R))) = \sum_{n,m\geq 1} n\, m\, c_n \overline{c_m} \iint_{D(0;R)} z^{n-1} (\overline{z})^{m-1}\, dx\, dy$$

(since within its radius of convergence, a power series is uniformly convergent, $\sum$ and $\iint$ can be interchanged)

Now $$\iint_{D(0;R)} z^{n-1}(\overline{z})^{m-1}\, dx\, dy = \int_0^{2\pi}\int_0^R \rho^{n+m-2} e^{i(n-m)\theta} \rho\, d\rho\, d\theta$$

$$= 0 \text{ if } n \neq m \quad \left(\int_0^{2\pi} e^{ik\theta}\, d\theta = \begin{cases} 0 ; & k \neq 0 \\ 2\pi ; & k = 0 \end{cases} \right)$$

and when $n = m$, we get : $$2\pi \cdot \left[\frac{\rho^{2n}}{2n}\right]_0^R = \pi \cdot \frac{R^{2n}}{n}.$$

Hence, $$\boxed{\text{Area of } f(D(0;R)) = \pi \sum_{n=1}^{\infty} n \cdot |c_n|^2 R^{2n}}$$

(39.5) Example continued.

(i) If $c_n = 1 \ \forall n$ (i.e. $f(z) = \frac{1}{1-z}$) and $0 < R < 1$, we get

$$\text{Area of } f(D(0;R)) = \pi \sum_{n=1}^{\infty} n\cdot(R^2)^n .$$

Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$\overset{d/dx}{\rightsquigarrow} \sum_{n=1}^{\infty} n.x^{n-1} = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} n(R^2)^n = \frac{\pi R^2}{(1-R^2)^2}$$

$\to \infty$ as $R \to 1$ (as $z=1$ is a pole of $\frac{1}{1-z}$ - we get an unbounded region).

(ii) $f(z) = \frac{1}{z}$; $D(\alpha; R) \to \mathbb{C}$. Let us assume that $|\alpha| > R$ so that the pole of $f(z)$, $0 \notin D(\alpha;R)$.

Area $(f(D(\alpha;R)))$ can be computed in two different ways.

Method 1: using the fact (see Problem Set 4, Problem 4(i)) that $z \mapsto \frac{1}{z}$ maps the circle $C(\alpha;R)$ to $C\left(\frac{\bar{\alpha}}{|\alpha|^2-R^2} ; \frac{R}{|\alpha|^2-R^2}\right)$

we get the area $= \pi \frac{R^2}{(|\alpha|^2-R^2)^2}$.

Method 2:
$$\frac{1}{z} = \frac{1}{\alpha + (z-\alpha)} = \frac{1}{\alpha} \cdot \frac{1}{1+\frac{z-\alpha}{\alpha}}$$
$$= \sum_{n=0}^{\infty} (-1)^n \alpha^{-n-1} (z-\alpha)^n \quad \left(\text{for } z \text{ s.t. } |z-\alpha| < |\alpha|\right)$$

Set $c_n = (-1)^n \alpha^{-n-1}$ in the formula given in 39.4 above to get

$$\text{Area} = \pi \sum_{n=1}^{\infty} n \cdot |\alpha|^{-2n-2} \cdot R^{2n}$$

$$= \frac{\pi R^2}{(|\alpha|^2 - R^2)^2}.$$

(39.6) Converse of Lemma (39.1) [Optional]

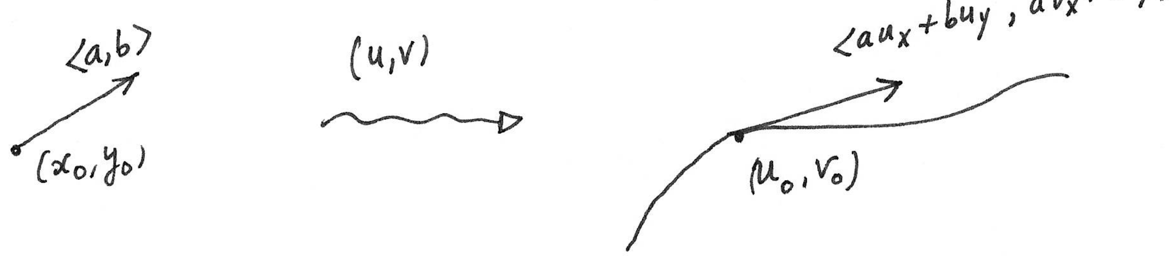
Let $\Omega \subset \mathbb{R}^2$ be an open, connected set ; $u, v : \Omega \to \mathbb{R}$ two functions in $C^1(\Omega; \mathbb{R})$ such that at $(x_0, y_0) \in \Omega$ we have

(i) The Jacobian $u_x v_y - u_y v_x > 0$

(ii) $(u,v) : \Omega \to \mathbb{R}^2$ is angle-preserving at (x_0, y_0).

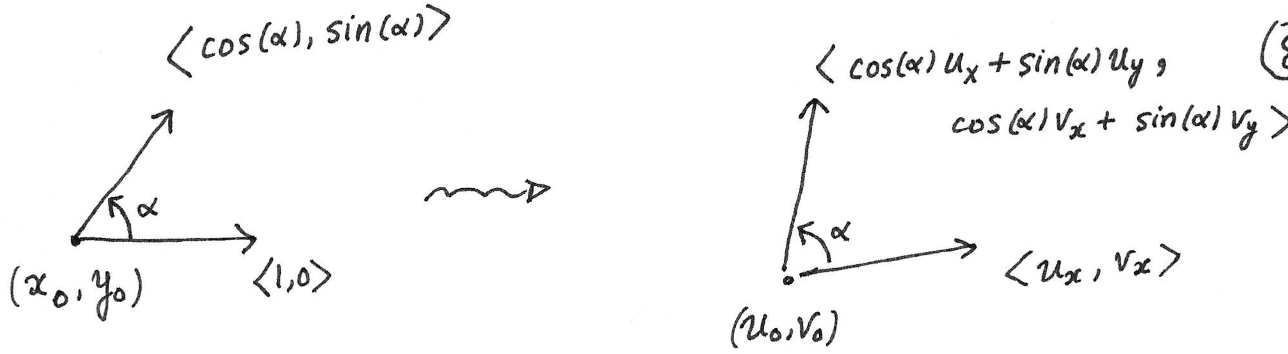
Then $u_x = v_y$ and $u_y = -v_x$ (Cauchy-Riemann eq^n's hold) at (x_0, y_0).

Proof: Note that for any vector $\langle a, b \rangle$, the tangent vector to the curve obtained by applying (u,v) to the line $(x_0 + at, y_0 + bt)$; at $t=0$, is $\langle a u_x + b u_y, a v_x + b v_y \rangle$



- Take a pair of orthogonal vectors $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$ at (x_0, y_0) to get

$\langle u_x, v_x \rangle \perp \langle u_y, v_y \rangle$ i.e. $u_x u_y + v_x v_y = 0$ — (1)



i.e. $$\cos(\alpha) = \frac{\langle u_x , v_x\rangle \cdot \langle \cos(\alpha) u_x + \sin(\alpha) u_y , \cos(\alpha) v_x + \sin(\alpha) v_y\rangle}{|\langle u_x , v_x\rangle| \cdot |\langle \cos(\alpha) u_x + \sin(\alpha) u_y , \cos(\alpha) v_x + \sin(\alpha) v_y\rangle|}$$

$$= \frac{\cos(\alpha)(u_x^2 + v_x^2) + \sin(\alpha)\overset{0}{(u_x u_y + v_x v_y)}}{\sqrt{u_x^2 + \cancel{u_y^2}\,v_x^2}\ \sqrt{\cos^2(\alpha)(u_x^2 + \cancel{u_y^2}\,v_x^2) + \sin^2(\alpha)(u_y^2 + v_y^2) + \sin(2\alpha)\underbrace{(u_x u_y + v_x v_y)}_{0 \text{ by } (1)}}}$$

$$\Rightarrow \quad u_x^2 + v_x^2 = u_y^2 + v_y^2 \quad - \ (2)$$

Combining (1) and (2) we get: either $u_x = v_y$ and $u_y = -v_x$

or $u_x = -v_y$ and $u_y = v_x$

The second possibility contradicts the assumption $u_x v_y - u_y v_x > 0$

Hence we obtain the Cauchy-Riemann equations. $\square$