Lecture 40

(40.0) Review of some important properties of holomorphic functions – useful for geometric considerations.

1. Cauchy-Riemann equations, harmonic functions.

$f:\Omega \to \mathbb{C}$ holomorphic, $\Omega \subset \mathbb{C}$, open, connected set.

$f(z) = u(x,y) + i\,v(x,y)$ $\quad z = x+iy \in \Omega$. Then.

(a) u, v are analytic (i.e., admit Taylor series expansions) solutions of

$$u_x = v_y \qquad \text{Cauchy-Riemann eq}^{\underline{n}}\text{s} \quad \left[\begin{array}{l}\text{See §8.7, Lecture 9.}\\ \text{§9.1}\end{array}\right]$$
$$u_y = -v_x$$

(b) Both u and v are harmonic – i.e. $\boxed{u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}}$ Laplace equation

(c) Conversely given $u \in C^2(\Omega)$ solution of Laplace equation and assuming Ω is simply-connected, $\exists!$ (upto constant) $v:\Omega \to \mathbb{R}$ s.t. $f = u + iv$ is holomorphic

In particular $u:\Omega \to \mathbb{R}$ is analytic.

[See §9.7, Lecture 10, §13.5, 14.1]

2. Open mapping and inverse function theorems.

(a) $\underset{\text{holomorphic}}{f:\Omega \to \mathbb{C}}$; $z_0 \in \Omega$, $f'(z_0) \neq 0$. Then $\exists\, \rho_1, \rho_2 > 0$ s.t.

$f: D(z_0,\rho_1) \xrightarrow{\sim} D(f(z_0),\rho_2)$ admits holomorphic inverse.

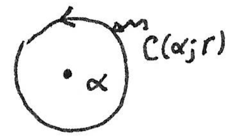
(b) In general, $f(\Omega) \subset \mathbb{C}$ is open – i.e. for every $z_0 \in \Omega$, $\exists\, \rho > 0$ s.t. $D(f(z_0),\rho) \subset f(\Omega)$.

[see Lectures 23, 24 – § 24.1, 24.2, 24.3]

3. Casorati - Weierstrass' Theorem. Let $f : \Omega \dashrightarrow \mathbb{C}$ be a meromorphic function. $z_0 \in \Omega$ an essential singularity. Then for every $r > 0$, $f(\mathcal{D}^{\times}(z_0; r)) \subset \mathbb{C}$ is dense. [§ 26·1]

4. Maximum modulus principle and mean value property. -

(a) $$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + R e^{i\theta}) \, d\theta$$



[§ 16·1]

($\overline{D(\alpha, r)} \subset \Omega$ - domain of f).

(b) $u : \Omega \to \mathbb{R}$ harmonic ; $\alpha \in \Omega$, $\alpha = (a, b)$.

$$u(a, b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + R\cos\theta, \, b + R\sin\theta) \, d\theta$$ [§ 16·1]

(c) For any contour $C \subset \Omega$ s.t. Interior $(C) = D \subset \Omega$; the maximum value of $|f(z)|$, $z \in \overline{D}$, occurs for $z \in C = \partial D$. [§ 16·2 - 4]. If $z_0 \in \Omega$ is a local max. for $|f|$, then f is constant near z_0. (hence constant by identity theorem - Lecture 22)

(d) $u \in C^2(\Omega; \mathbb{R})$ harmonic function takes its max. and min. values at the boundary - otherwise it is a constant. [§ 16·2 - 16·4].

5. Argument principle. - $$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = -\text{Number of poles of } f \text{ within } \gamma + \text{Number of zeroes of } f \text{ within } \gamma$$

(counted with multiplicities)

[Lecture 29. - § 29.3 , 29.4]

(40.1) Some more terminology related to conformal geometry

(i) Conformal map = angle preserving (in the oriented sense)

= holomorphic functions $f : \Omega \to \mathbb{C}$ s.t. $f'(z) \neq 0$ $\forall z \in \Omega$.

[see last lecture]

(ii) Univalent = one-one $(f(z_1) = f(z_2) \Rightarrow z_1 = z_2)$.

Note: univalent $\Rightarrow$ conformal, but not conversely

(recall - if f is s.t. $f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$; $f^{(m)}(z_0) \neq 0$, then $\exists\, \rho_1, \rho_2 > 0$ s.t. $f : D(z_0, \rho_1) \to D(f(z_0), \rho_2)$ is m-to-one.)

e.g. $f(z) = e^z$; $f'(z) = e^z \neq 0$ $\forall z \in \mathbb{C}$. But $f = e^z : \mathbb{C} \to \mathbb{C}$ is not univalent. Note however

$$e^z : \{ 0 < \mathrm{Im}(z) < 2\pi \} \to \mathbb{C} \quad \text{is univalent.}$$

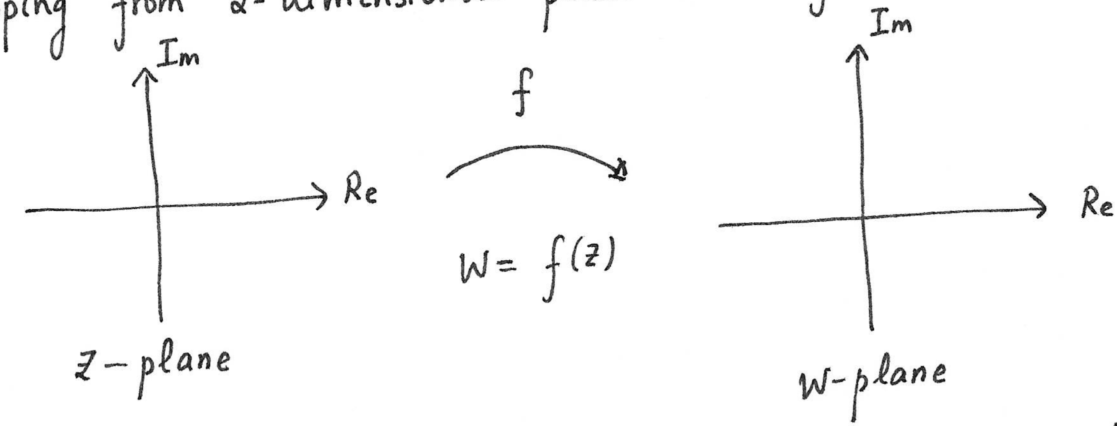
(iii) Conformal automorphism = biholomorphism : $f : \Omega \to \Omega'$ s.t. f admits a holomorphic inverse.

(equivalence) (when $\Omega = \Omega'$)

(40.2) Geometric interpretation of conformal equivalences.

Given a holomorphic function $f : \mathbb{C} \to \mathbb{C}$, there are a few methods of "visualizing" f - analoguous to drawing graph of a function $\mathbb{R} \to \mathbb{R}$.

- Mapping from 2-dimensional plane to itself.

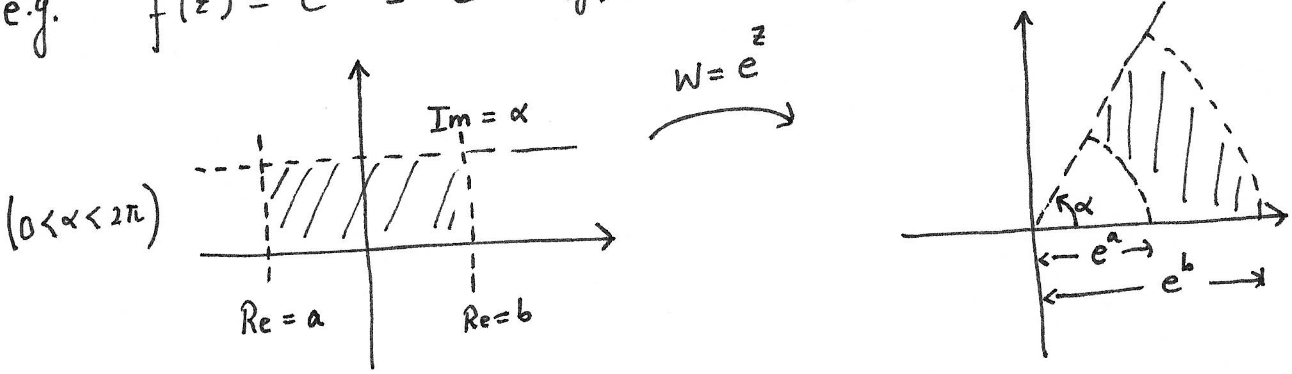


One often draws a few curves / regions in the z-plane and its image in the w-plane.

- Since, in principle, the information of f is contained in $u = \mathrm{Re}(f)$ (§40.0-(1)) - some mathematician prefer to sketch level curves of ~~f~~ $u : \mathbb{R}^2 \to \mathbb{R}$ (i.e. curves $u(x,y) = C$ constant)

Or, the vector field $\langle u_x(x,y), u_y(x,y) \rangle$. Recall that $f'(z) = u_x - i u_y$ - so this approach sketches $\overline{f'(z)}$.

e.g. $f(z) = e^z = e^x \cos(y) + i e^x \sin(y)$.



$u(x,y) = e^x \cos(y)$. Level curves : $e^x \cos(y) = C$

i.e. $x = \ln\left(\frac{C}{\cos(y)}\right)$

$\vec{\nabla} u = \langle u_x, u_y \rangle = \langle e^x \cos(y), -e^x \sin(y) \rangle$

(40.3) The following lemma is very useful in determining the image of a region under a holomorphic function, and determining whether it is univalent.

Lemma. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Assume that

- Ω = Interior of a contour C in $\mathbb{C}$
- f extends to a continuous $f : \overline{\Omega} \to \mathbb{C}$ and $f(C) = C'$ is one-one - and C' is again a contour. Let Ω' = Interior(C').

Then $f : \Omega \to \Omega'$ is conformal equivalence (i.e. one-one and onto holomorphic).

Proof.- For any $w_0 \in \mathbb{C}$, $w_0 \notin C'$, we have

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - w_0} \, dz = \# \{ z \in \Omega \mid f(z) = w_0 \}$$

$$= \frac{1}{2\pi i} \int_{C'} \frac{dw}{w - w_0} \quad \text{(change of variables } w = f(z))$$

$$= \begin{cases} 1 & \text{if } w_0 \in \text{Interior}(C') \\ 0 & \text{if } w_0 \notin \text{Interior}(C') \end{cases} \quad \text{(Cauchy's integral formula)}$$

$\Rightarrow f : \Omega \to \Omega'$ is 1-1 and onto. $\square$

(40.4) Lemma (40.3) and max. modulus principle (40.0–4) signify the importance of holomorphic / harmonic functions on the boundary of a region – which motivated the following.

Dirichlet's boundary value problem. Let C be a contour in $\mathbb{C}$,

$\Omega = \text{Interior}(C)$ (note: Ω is open, connected, simply-connected),

$\overline{\Omega} = \Omega \cup C$. Assume a continuous $g : C \to \mathbb{R}$ is given.

Find $u \in C^2(\Omega)$ harmonic s.t. u extends to a continuous $\overline{\Omega} \to \mathbb{R}$

and $u\Big|_{\partial\Omega = C} = g$.

(This is only a 2-dim'l version of the general Dirichlet BVP).

Dirichlet's principle. [Ω, C, g – as before]

Consider the set of all $\left\{ \begin{array}{l} f : \Omega \to \mathbb{R} \\ f \in C^2(\Omega) \end{array} \middle| \begin{array}{l} f \text{ extends to continuous } \overline{\Omega} \to \mathbb{R} \\ f\big|_{\partial\Omega} = g \end{array} \right\} =: \mathcal{F}$

Define $D(f) := \displaystyle\iint_{\Omega} (f_x^2 + f_y^2)\, dx\, dy$. Then $u : \Omega \to \mathbb{R}$ minimizing: $D(u)$

$\text{Min}\{D(f) : f \in \mathcal{F}\} = D(u)$ is the required harmonic function.

→ As stated, Dirichlet's principle assumes that $I : \mathcal{F} \to \mathbb{R}$ does attain its minimum at some $u \in \mathcal{F}$. This is false in general.

Lejeune (Johann Peter Gustav) Dirichlet (13/2/1805 – 5/5/1859)

(40.5) Uniqueness of the solution of Dirichlet's boundary value problem.

If $u_1, u_2 \in C^2(\Omega; \mathbb{R})$ are two harmonic functions s.t. both extend to continuous $\overline{\Omega} \to \mathbb{R}$ and $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$ then $u_1 - u_2$ is a harmonic function, identically zero on $\partial\Omega$. Since every harmonic function takes extremal values on $\partial\Omega$, we get $u_1 - u_2 = 0$ i.e. $u_1 = u_2$ on Ω.

(40.6) Variational problem. [Ω, C, g as before] [OPTIONAL]

$$\mathcal{F} = \left\{ w \in C^2(\Omega; \mathbb{R}) \;\middle|\; \begin{array}{l} w \text{ extends to a continuous function} \\ \overline{\Omega} \to \mathbb{R}, \text{ and } w\big|_{\partial\Omega} = g \end{array} \right\}$$

$$D : \mathcal{F} \to \mathbb{R}_{\geq 0} \qquad D(\psi) = \iint_{\Omega} (\psi_x^2 + \psi_y^2)\, dx\, dy \quad \text{(Dirichlet integral)}$$

The problem is to find $u \in \mathcal{F}$ such that $D(u) \leq D(w) \;\forall w \in \mathcal{F}$.

Lemma. If $u \in \mathcal{F}$ is a solution of the variational problem stated above, then u is harmonic (hence ~~a~~ the solution of Dirichlet's boundary value problem.

Proof. The argument is a standard trick in calculus of variations. Let $\varphi \in C^2(\Omega; \mathbb{R})$ s.t. φ extends to a continuous $\overline{\Omega} \to \mathbb{R}$ and $\varphi\big|_{\partial\Omega} = 0$. Thus $u + \varphi \in \mathcal{F}$ and hence

$$D(u+\varphi) - D(u) \geq 0.$$

$$D(u+\varphi) - D(u) = D(\varphi) + 2 \iint_{\Omega} (u_x \varphi_x + u_y \varphi_y)\, dx\, dy.$$

Using Green's Theorem:

$$\int_{\partial\Omega} (u_x \varphi)\, dy - (u_y \varphi)\, dx = \iint_{\Omega} \Big((u_{xx} + u_{yy})\varphi + (u_x \varphi_x + u_y \varphi_y) \Big)\, dx\, dy.$$

As $\varphi\big|_{\partial\Omega} = 0$, we get

$$D(u+\varphi) - D(u) = D(\varphi) - 2 \iint_{\Omega} (u_{xx} + u_{yy})\varphi\, dx\, dy \geq 0$$

$$\forall\, \varphi \in C^2(\Omega; \mathbb{R}), \quad \varphi\big|_{\partial\Omega} = 0$$

Using the existence of "bump functions"

($\forall\, z_0 \in \Omega$ and $r_1 < r_2$ $(r_1, r_2 > 0)$ ~~and~~ such that $\overline{D(z_0; r_2)} \subset \Omega$,

$\exists\, \varphi \in C^2(\Omega; \mathbb{R})$ s.t. $\varphi(z) = 1$ on $D(z_0; r_1)$

$= 0$ on $\Omega \setminus D(z_0; r_2)$)

one concludes that $u_{xx} + u_{yy} = 0$. $\square$

(40.7)* Riemann's original incorrect argument for mapping theorem (1851) was based on Dirichlet's principle and is given below.

Riemann Mapping Theorem - Version 1. - Let Ω, C be as before. Then $\exists$ a conformal equivalence $f : \Omega \to \mathbb{D}$.

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* Optional

"Proof": Let $z_0 \in \Omega$ (we want to map z_0 to $0 \in \mathbb{D}$).

Consider $g : \partial\Omega \to \mathbb{R}$ given by

$$g(z) = \ln\left(\frac{1}{|z-z_0|}\right).$$

Let $u \in C^2(\Omega; \mathbb{R})$ be harmonic s.t. u extends to $\overline{\Omega} \to \mathbb{R}$ and $u\big|_{\partial\Omega} = g$ (using Dirichlet's principle). Let $v \in C^2(\Omega; \mathbb{R})$ be its harmonic conjugate (exists since Ω is simply-connected).

Define $f(z) = (z - z_0)\exp(u(z) + iv(z)) : \Omega \to \mathbb{C}$.

Note: $f(z_0) = 0$ and for $z \in \partial\Omega$, $|f(z)| = |z - z_0| \cdot e^{g(z)} = 1$

Hence f maps Ω to $\mathbb{D}$.