Lecture 41

Conformal automorphisms of complex plane and Riemann sphere.

(41.0) Recall: for an open, connected set $\Omega \subset \mathbb{C}$,

$$\mathrm{Aut}(\Omega) = \{ f : \Omega \to \Omega \text{ holomorphic bijection} \}$$

Proposition. If $f \in \mathrm{Aut}(\mathbb{C})$, then $\exists\, a, b \in \mathbb{C}$, $a \neq 0$ s.t.

$$f(z) = az + b \quad \forall z \in \mathbb{C}.$$

Proof. $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function (and a bijection).

So, ∞ is an isolated singularity of f.

<u>Claim:</u> f does not have an essential singularity at ∞.

(Proof of the claim.- assume, for the sake of a contradiction, that f has an essential singularity at ∞. Then, by Casorati-Weierstrass theorem [See §26.1], $\forall R > 0$, $f(\mathbb{C} \setminus \overline{D(0;R)}) \subset \mathbb{C}$ is a dense set - i.e., $\forall w_0 \in \mathbb{C}$ and $\rho > 0$; $\exists\, z_0 \in \mathbb{C}$, $|z_0| > R$ s.t.

$$|f(z_0) - w_0| < \rho.$$

By open mapping theorem, $f(D(0;1))$ is an open set. In particular, there exists $r > 0$ s.t. $D(f(0); r) \subset f(D(0;1))$.

Take $R > 1$ in the Casorati-Weierstrass theorem to get $z_0 \in \mathbb{C}$, $|z_0| > 1$ s.t. $f(z_0) \in D(f(0); r) \subset f(D(0;1))$

i.e. $\exists\, z_1 \in D(0;1)$ s.t. $f(z_0) = f(z_1)$. But $z_0 \neq z_1$ $\left(\begin{array}{l} |z_0| > 1 \\ |z_1| < 1 \end{array} \right)$

contradicting univalence (1-1 ness) of f.)

Therefore, ∞ is either a removable singularity or a pole of order $n \in \mathbb{Z}_{\geq 1}$ of f.

If removable, f would be a constant - again a contradiction to f being a bijection.
[See § 26.4]

If ∞ is a pole of order n, then f is a polynomial of degree n, implying that f is n-to-1 map. Hence $n = 1$ - i.e., f is of the form $az+b$. □

(41.1) Möbius transformations - also called linear fractional transformations.

Given $a, b, c, d \in \mathbb{C}$; s.t. $ad - bc \neq 0$ $\quad z \mapsto \dfrac{az+b}{cz+d}$ is called a Möbius transformation. The calculation given below

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{cz+d} \quad \text{if } c \neq 0$$

$$= \frac{a}{d} z + \frac{b}{d} \quad \text{if } c = 0$$

shows that Möbius transformation is a composition of three elementary ones:

Sealing : $\sigma_p(z) = p \cdot z \quad (p \neq 0)$

Translation : $\tau_x(z) = z + x \quad (x \in \mathbb{C})$

Inversion : $I(z) = \dfrac{1}{z}$

Notation : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2x2 invertible matrix

$$M_A(z) = \frac{az+b}{cz+d}$$ Möbius transformation.

(41.2) Lemma. - $M_{A_1} \circ M_{A_2} = M_{A_1 A_2}$.

Proof. - Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Then $M_{A_1}(M_{A_2}(z)) = M_{A_1}\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right)$

$$= \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2) z + c_1 b_2 + d_1 d_2} = M_{A_1 A_2}$$ □

Hence, M_A is invertible ($M_A \circ M_{A^{-1}} = M_{Id}$; and $Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{Id}(z) = z \quad \forall z \in \mathbb{C}$).

Note : $M_A : \mathbb{C} \dashrightarrow \mathbb{C}$ has a simple pole at $\frac{-d}{c}$ if $c \neq 0$;

If $c = 0$; $M_A : \mathbb{C} \to \mathbb{C}$ has a simple pole at ∞.

(41.3) Möbius transformations preserve circles and lines in the complex plane.

This is true for each of the elementary transformations [Problem 4 of Set 4] (scaling, translation and inversion), hence for any Möbius transformation by §41.1. For future reference, let us write the effect of each of the elementary transformation on a typical circle / line.

For $\alpha \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$; $C(\alpha; r) = \{ z \mid |z-\alpha| = r \}$ circle of radius r, centered at α.

For $w \in \mathbb{C}$, $a \in \mathbb{R}$; $L(w; a) = \{ z \mid \mathrm{Re}(zw) = a \}$
$(w \neq 0)$

line with equation $Ax + By = a$ where $w = A - Bi$.

<u>Translation</u> $(x \in \mathbb{C})$:

$$\tau_x(C(\alpha; r)) = C(\alpha + x; r)$$

$$\tau_x(L(w; a)) = L(w; a + \mathrm{Re}(xw))$$

<u>Scaling</u> $p \in \mathbb{C} \setminus \{0\}$

$$\sigma_p(C(\alpha; r)) = C(p\alpha; |p| r)$$

$$\sigma_p(L(w; a)) = L\left(\frac{w}{p}; a\right)$$

<u>Inversion</u>

$$I(C(\alpha, r)) = \begin{cases} L(\alpha; \frac{1}{2}) & \text{if } |\alpha| = r \text{ (i.e. } 0 \in C(\alpha; r)) \\ C\left(\dfrac{\bar{\alpha}}{|\alpha|^2 - r^2}; \dfrac{r}{\left| |\alpha|^2 - r^2 \right|}\right) & \text{if } |\alpha| \neq r \end{cases}$$

$$I(L(w;a)) = \begin{cases} L(\overline{w};0) & \text{if } a = 0 \\ C(w;|w|) & \text{if } a = \frac{1}{2} \end{cases}$$

$\leftarrow$ i.e. $a \neq 0$ which can be scaled

(Note : if $a \neq 0$; $L(w;a) = L\left(\frac{w}{2a};\frac{1}{2}\right)$).

(41.4) Möbius transformations are often viewed as automorphisms of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \quad M_A(z) = \frac{az+b}{cz+d}$$

- if $c \neq 0$; $M_A\left(-\frac{d}{c}\right) = \infty$

  $M_A(\infty) = \frac{a}{c}$

- if $c = 0$; $M_A : \mathbb{C} \to \mathbb{C}$ and $M_A(\infty) = \infty$.

Viewing lines in the complex plane as circles on $\hat{\mathbb{C}}$ passing through ∞, the result of § 41.3 is stated simply as :

Möbius transformations map circles to circles (on $\hat{\mathbb{C}}$).

(41.5) More precise form of the assertion from the last paragraph -

By an (conformal) automorphism of $\hat{\mathbb{C}}$, we mean a bijective function

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that f restricted to $\mathbb{C}$ is a meromorphic function. We claim that any such f is necessarily of the form M_A for some invertible $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a bijection s.t. $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic.

- If $f(\infty) = \infty$, then $f : \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$, hence of the form $f(z) = az + b$ by Prop. 41.0 above.
  i.e. $f(z) = M_A(z)$ for $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ $(a \neq 0)$.
- If $f(\infty) \neq \infty$, then $\exists!\ \beta \in \mathbb{C}$ s.t. ~~$f(\beta) = \infty$~~ $f(\infty) = \beta$.
  Consider the Möbius transformation $z \mapsto \dfrac{1}{z - \beta}$
  Then $M_A \circ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (i.e. M_A for $A = \begin{bmatrix} 0 & 1 \\ 1 & -\beta \end{bmatrix}$)
  is again a bijection sending ∞ to ∞.
  $\Rightarrow M_A \circ f = M_B \quad \Rightarrow f = M_{A^{-1}B}$ is a Möbius transformation $\square$

In other words, $\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ bijection} \,\middle|\, \begin{array}{l} f \text{ restricted to} \\ \mathbb{C} \text{ is meromorphic} \end{array} \right\}$

$$= \{ M_A : A \in \underbrace{GL_2(\mathbb{C})}_{} \}$$

↳ group of 2×2 invertible matrices.